Finding a Bounded-Degree **Expander** Inside a Dense One

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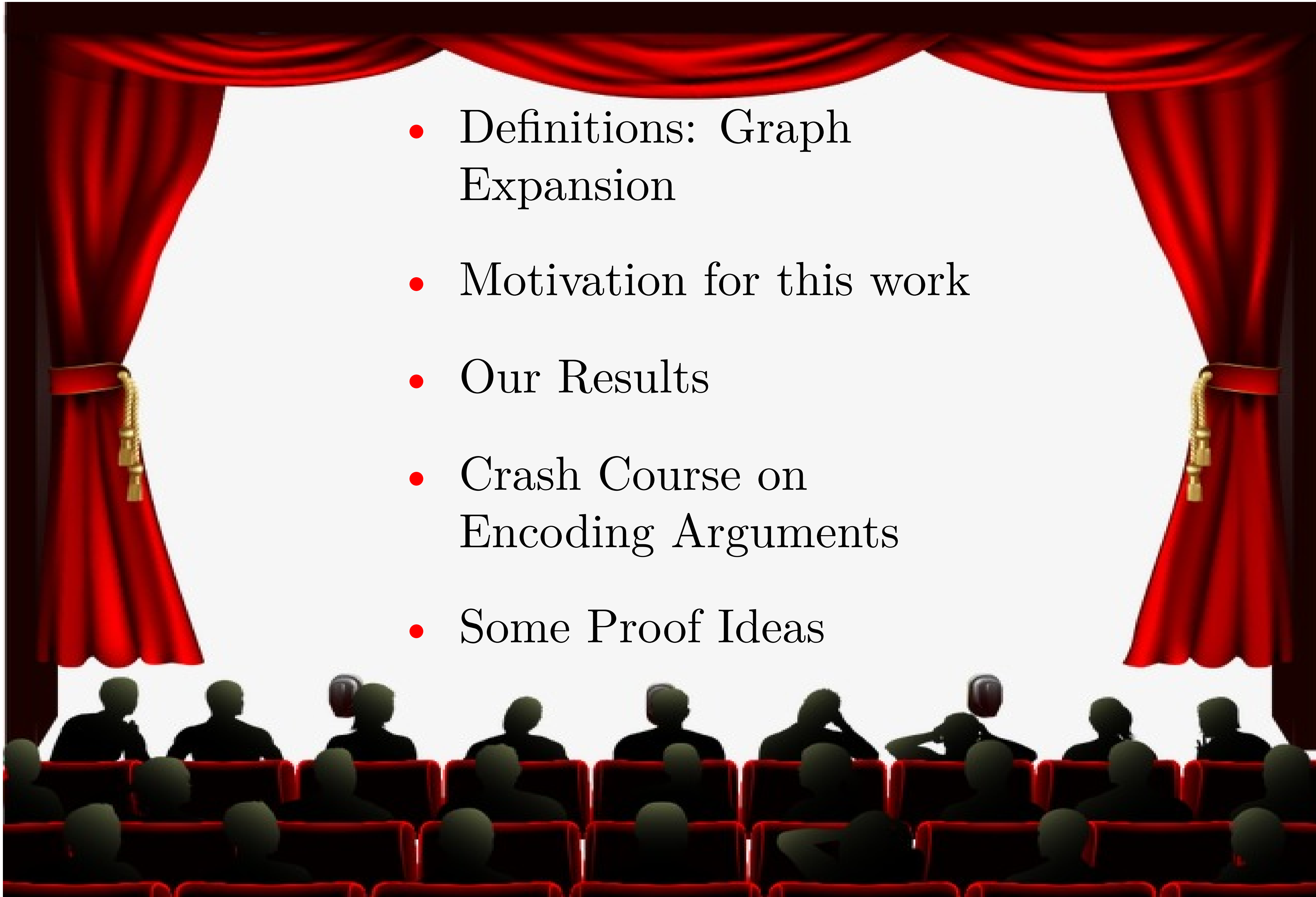
COATI Group Seminar
26 March 2019

Outline



- Definitions: Graph Expansion
- Motivation for this work
- Our Results
- Crash Course on Encoding Arguments
- Some Proof Ideas

Graph Expansion I

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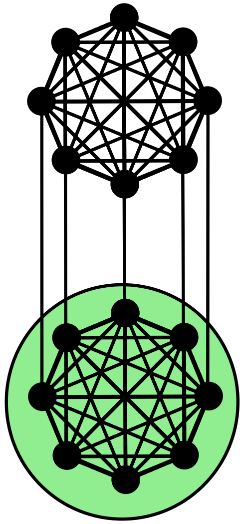

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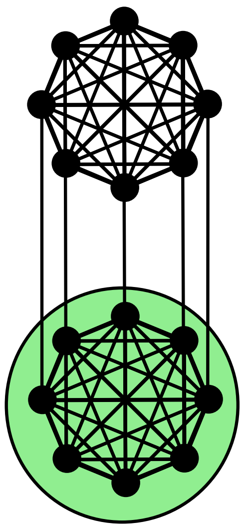

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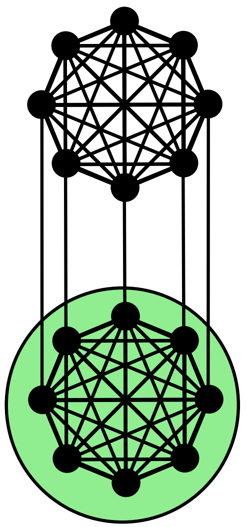

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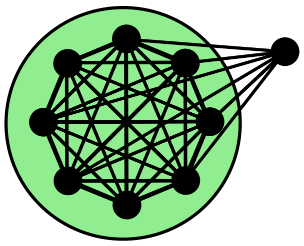

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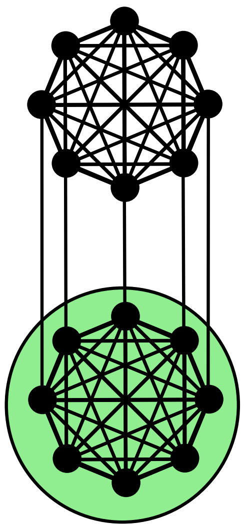

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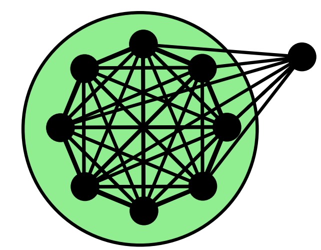

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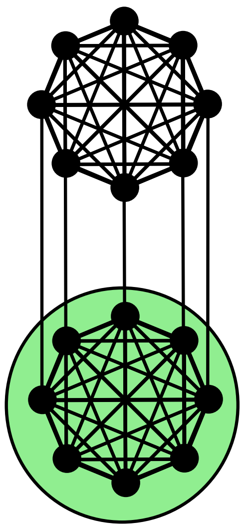

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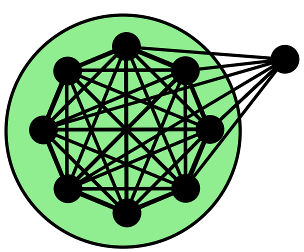

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In regular graphs $\frac{e(S,V-S)}{\min\{vol(S),vol(V-S)\}}$ is equivalent to $\phi(S) = \frac{e(S,V-S)}{vol(S)}$ assuming $S \leq \frac{n}{2}$

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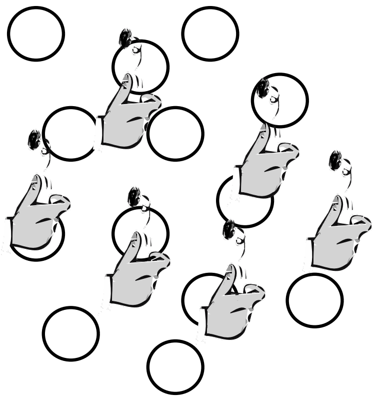
Example:

In an Erős-Rényi graph $G_{n,p}$, include each edge with prob p.



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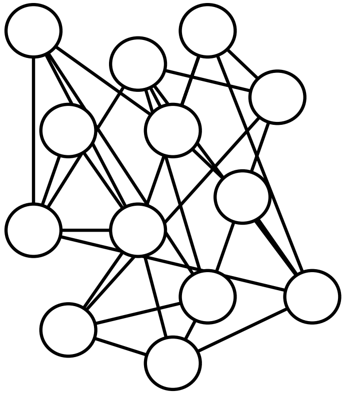
For any $p \gg \frac{\log n}{n}$, they are good expanders with high probability.



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$$2e(S,S) - \frac{\Delta}{n}|S|^2$$
$$= 1_S^T(A - \frac{\Delta}{n}J)1_S$$
$$\leq \lambda||1_S||^2 = \lambda|S|$$

2nd-largest eigenvalue

Motivations for this Work I

Distributed construction of constant-degree expanders

Corollary of Marcus-Spielman-Srivastava proof's of the Kadison-Singer conjecture [Ann. of Math. '15]:

Every dense expander has a *constant-degree subgraph* which is also an expander.

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Corollary of Marcus-Spielman-Srivastava proof's of the Kadison-Singer conjecture [Ann. of Math. '15]:

Every dense expander has a *constant-degree subgraph* which is also an expander.

But the proof is non-constructive:
How to find the *low-degree sub-expander*?

Motivations for this Work II

Several works propose complicated distributed construction of expanders:

- Law and Siu [INFOCOM'03]: incremental construction using Hamiltonian cycles

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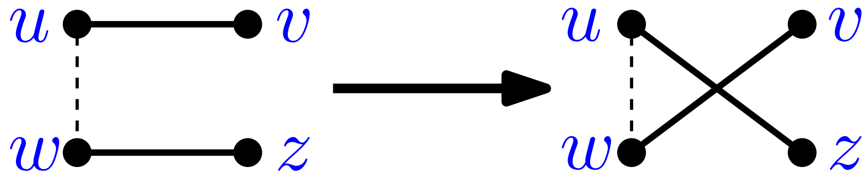
Several works propose complicated distributed construction of expanders:

- Law and Siu [INFOCOM'03]: incremental construction using Hamiltonian cycles
- Allen-Zhu et al. [SODA'16]: start with a $\Omega(\log n)$-regular graph and increase its expansion



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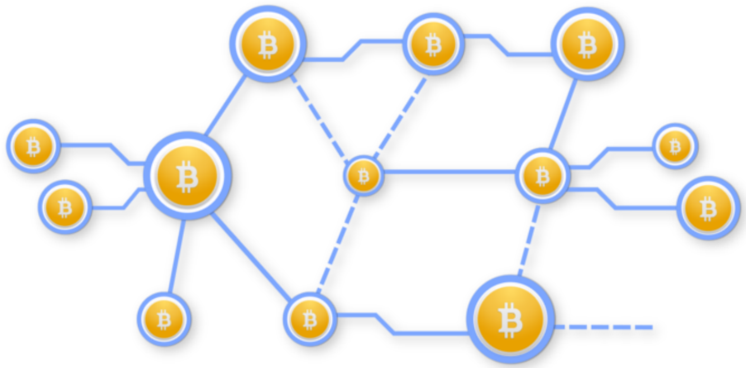

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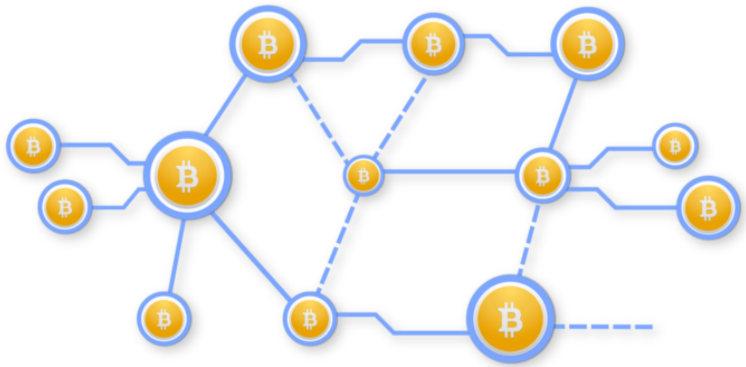
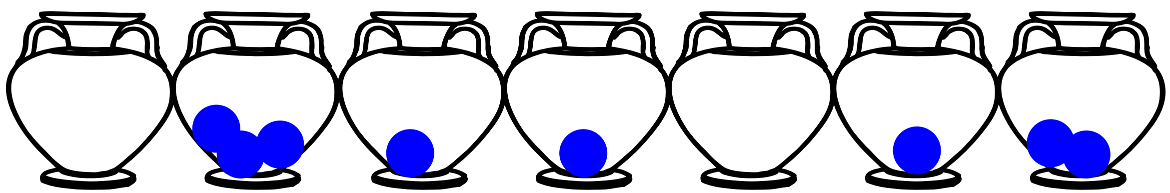


- Distributed construction of constant-degree graph implies *constant-load balancing* algorithm.
  Previous works: almost-tight load balancing in poly time (Berenbrink et al., SPAA'14)



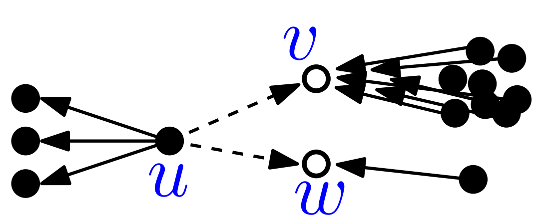
Algorithm Request - Accept if Enough Space

Algorithm RAES(G, d, c) for each node v:

- Set $d_{out} = 0$ and assume connections are directed
- At the start of each round,
  if ($d_{out} < d$) then
  send $d - d_{out}$ requests to random neighbors
- At the end of each round
  if (current requests $+$ new ones $\leq cd$) then
  accept all request
  else
  reject all current requests

  if ($d_out = d$) then
  forget edge orientation

Example with $d = 5$



u is missing 2 connections.
u asks to connect to w and v.
v has already cd incoming connections and refuses u's requests.

Our Result

Theorem.
For every $d \gg 1$, $0 < \alpha \leq 1$, $c \gg \frac{1}{\alpha^2}$, and αn-regular graph G, w.h.p.
RAES(G, d, c) *runs in* $\mathcal{O}(\log n)$ *parallel rounds with message complexity is* $\mathcal{O}(n)$*.*
Moreover, if G's 2nd-largest eigenvalue λ of normalized adjacency matrix is $\leq \epsilon\alpha^2$, then w.h.p.
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Proof Technique: *Encoding Argument*
(omitted: message complexity using martingale theory)

Encoding Arguments

Encoding Lemma.
If X finite set and
$C : X \to \{0,1\}^*$ a (partial & prefix-free) encoding of X then

$$\Pr_{x \sim Unif(X)}(|C(x)| \leq \log|X| - s) \leq 2^{-s}$$

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Proof. $\frac{2^{\log|X|-s}}{|X|} \le 2^{-s}$.

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Suggested reading: P. Morin et al. *Encoding Arguments*, ACM Comp. Surveys '17.

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Call B a *bad substring* of $\log n + s$ consecutive heads.
Consider encoding C_B for strings containing B:

(index i of first bit of B , all other bits of the string except those at entry $i, i+1, \ldots, i+\log n+s$)

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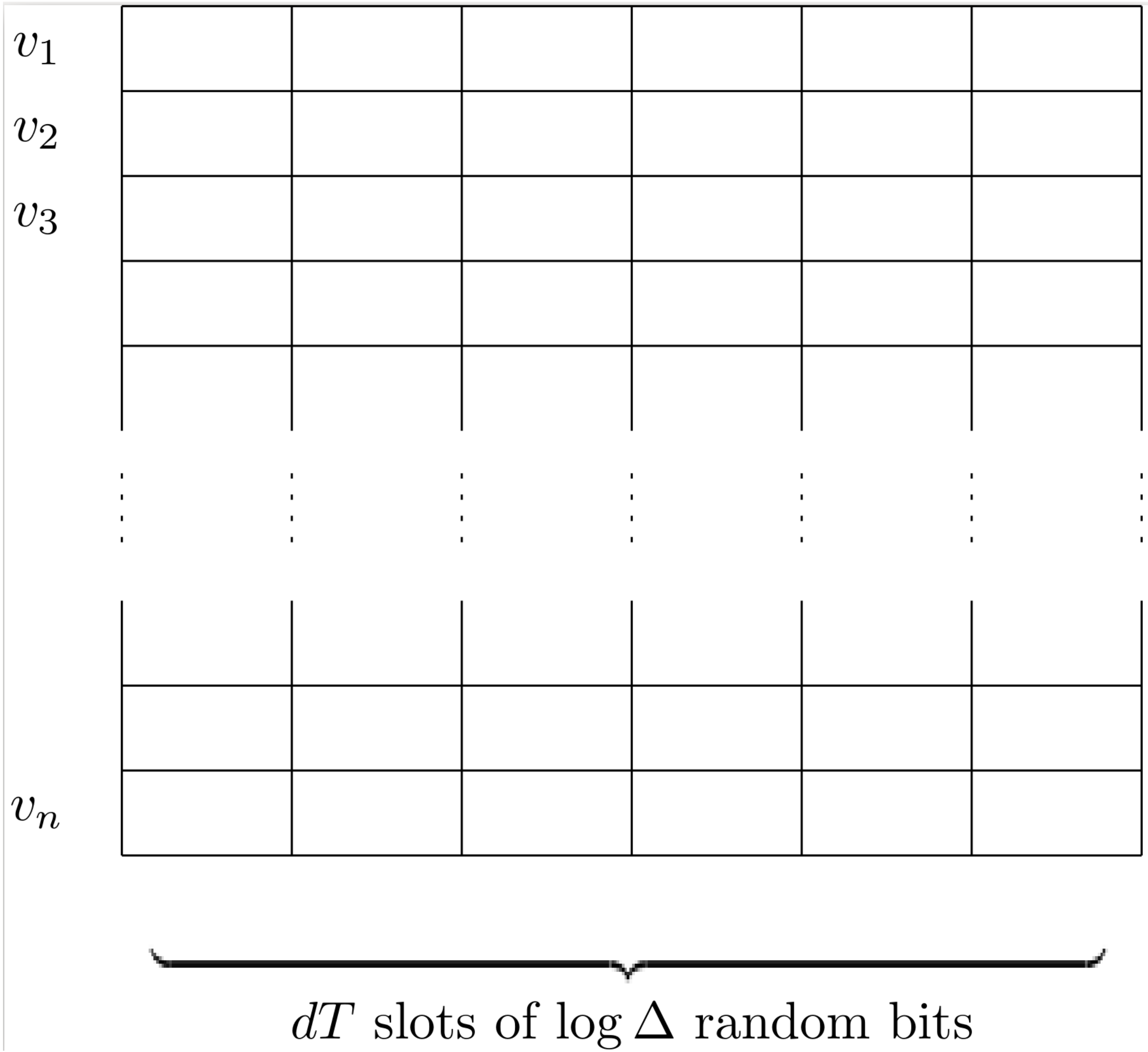
By the Encoding Lemma
$\Pr(|C_B(x)| \le \log|X| - s) = \Pr(|C_B(x)| \le n - s) \le 2^{-s}$

Encoding Arg. for Running Time (Warm Up)

Implementation: For each node v_i, array of dT entries of $\log \Delta$ bits

If RAES doesn't terminate in $O(\log n)$ rounds there exist node v with a rejected request at each round



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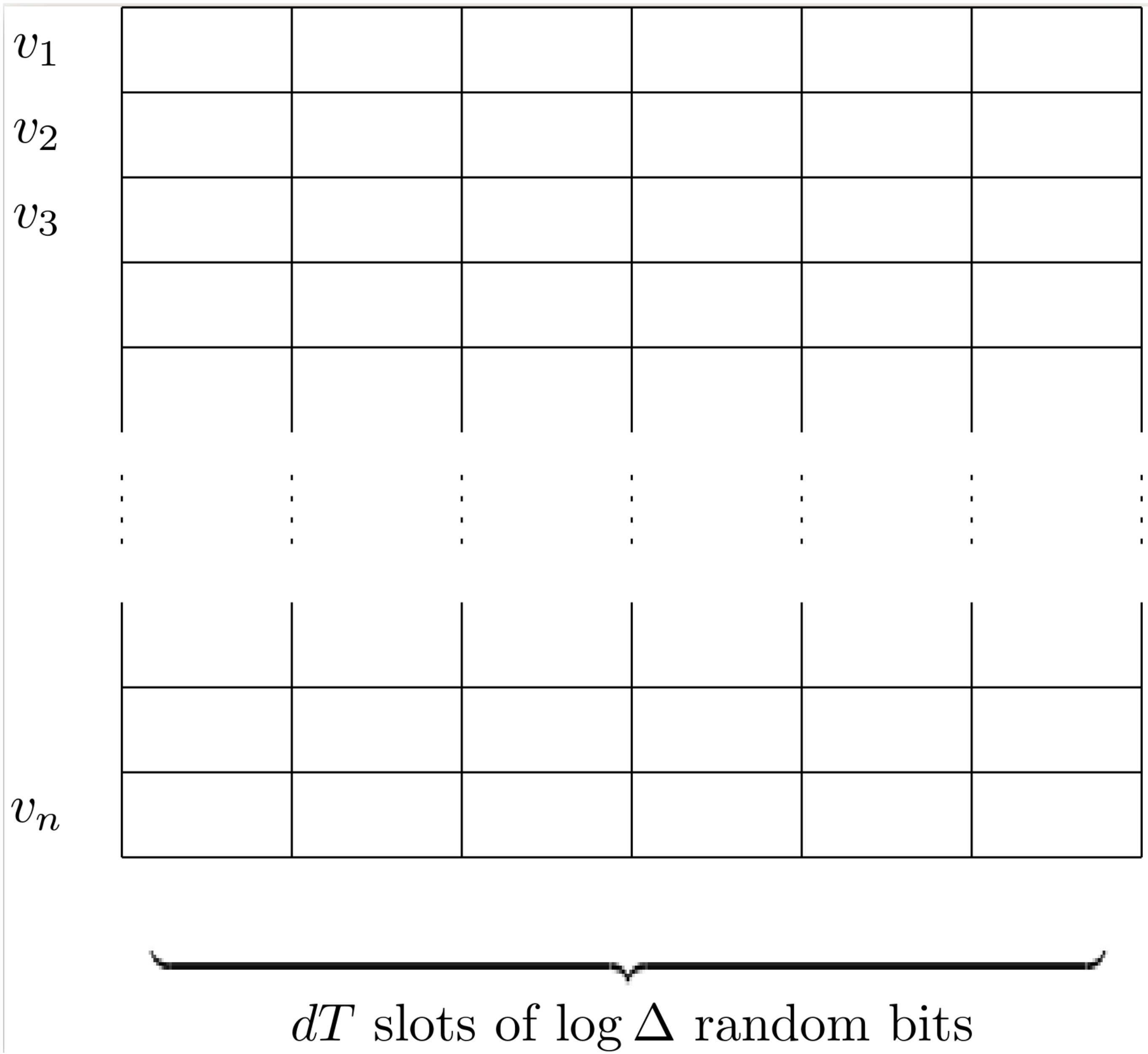
After calculations we see that we save $\frac{1}{2}\ell_v \log(\alpha c) - \log n = \Omega(\log n)$

Encoding Argument for Expansion

Implementation: For each node v_i, array of dT entries of $\log \Delta$ bits

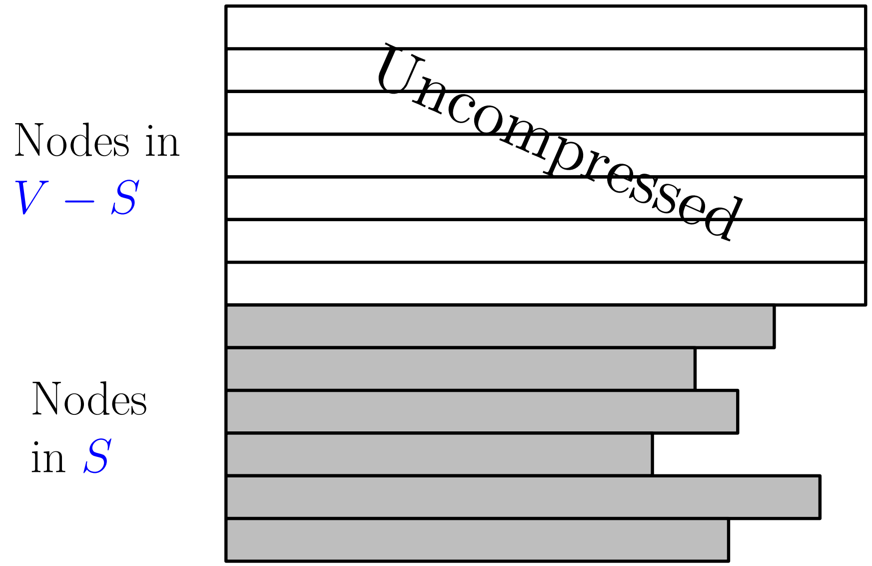
We show that if the execution results in a non-expander, then it can be represented with $ndt \log \Delta - \Omega(\log n)$ bits



Compressing the Non-Expanding Set

Encoding:

- Randomness of $V - S$
- Set S: $\log|S| + \log\binom{n}{s}$



Compressing the Non-Expanding Set

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- Randomness of $V - S$
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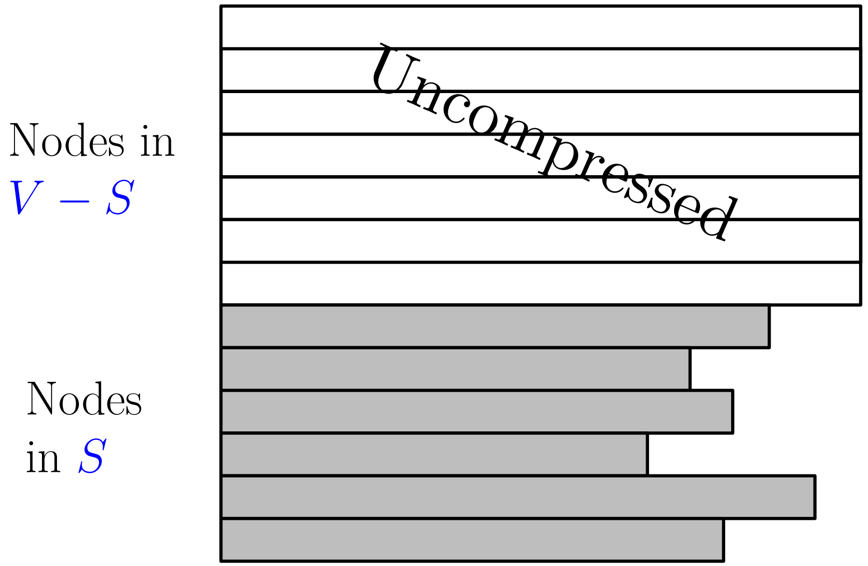
- Accepted connections: $\sum_{v \in S} 2\log \ell_v + \log\binom{\ell_v}{d}$



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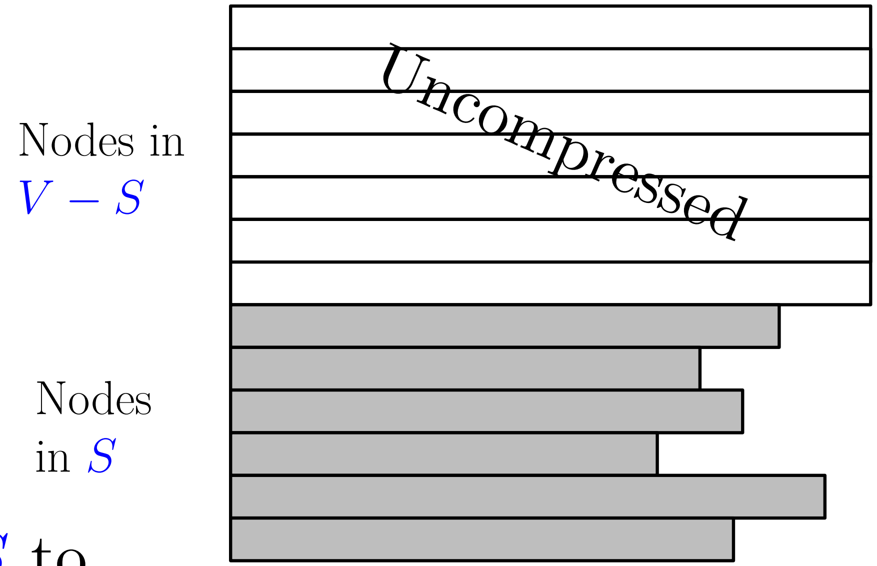
- Accepted connections from S to $V - S$: $\sum_{v \in S} 2 \log(\epsilon_v d) + \log \binom{d}{\epsilon_v d}$

ϵ_v: fraction of v's accepted connections towards $V - S$



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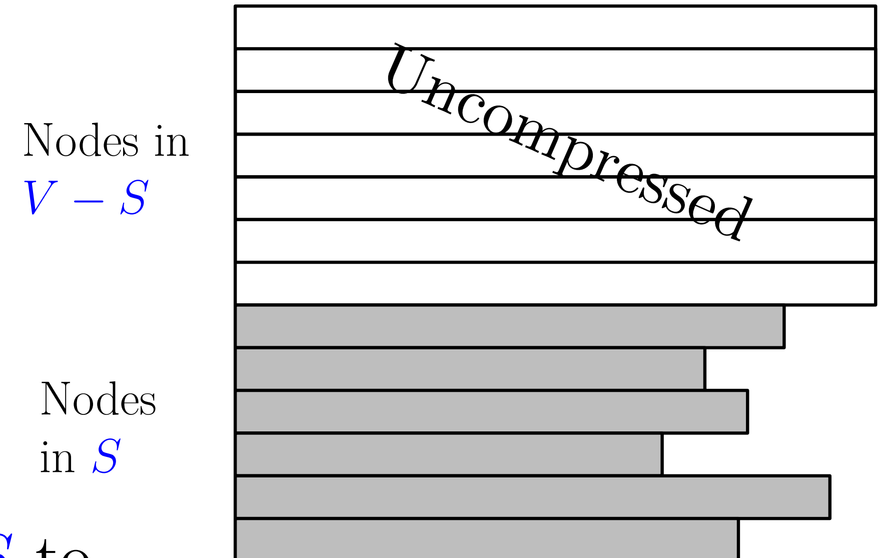

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- Destinations of connections from S: $\sum_{v \in S} (1 - \epsilon_v) d \log((1 - \delta_v)\Delta) + \sum_{v \in S} \epsilon_v d \log \Delta$

  connections to S — connections to $V - S$ (uncompressed)

δ_v: fraction of v's edges towards $V - S$ in G

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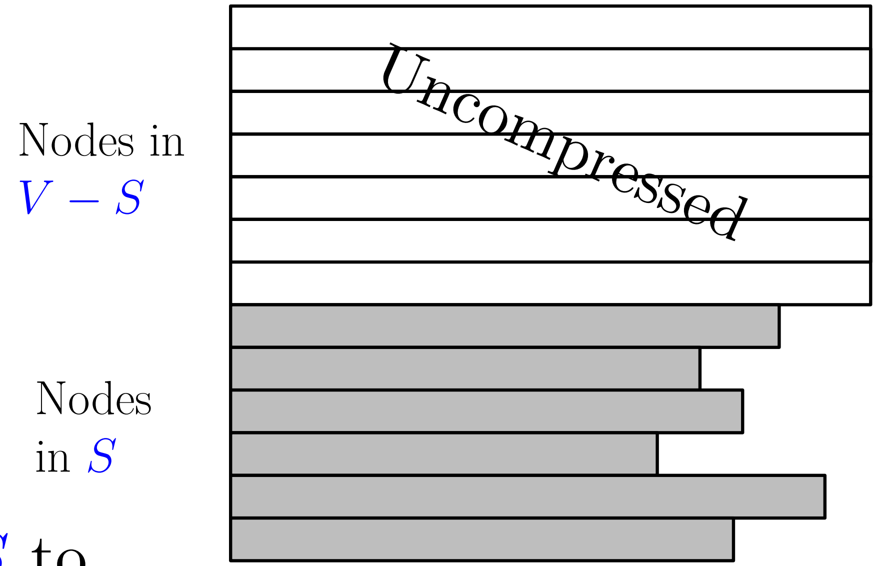
- **Rejected requests**



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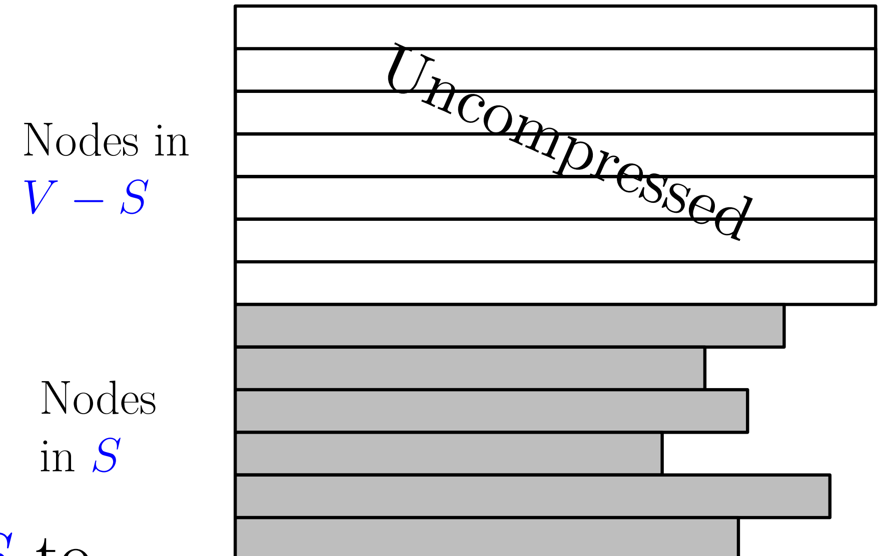

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- **Rejected requests**
- Unused randomness (after node's termination)

Compressing the Non-Expanding Set

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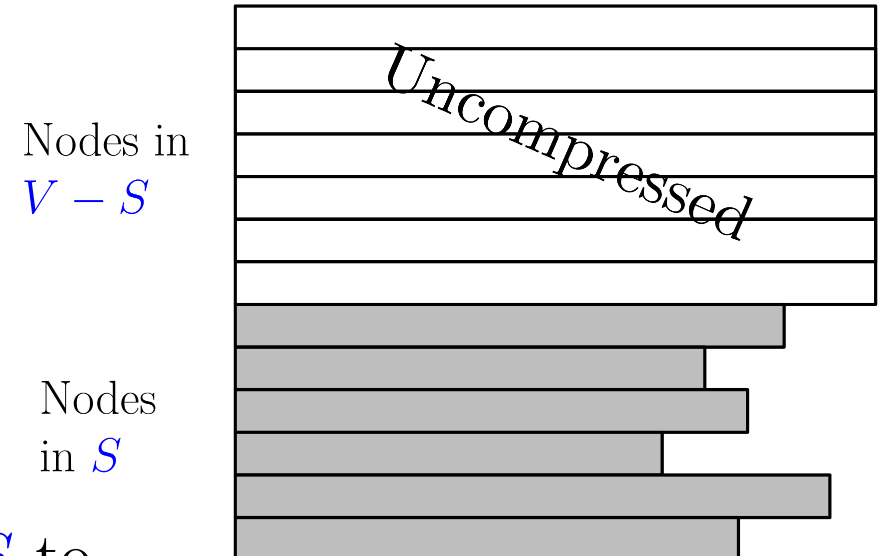

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To represent accepted requests from S we need

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$$\leq sd \log \Delta - \frac{1 - \epsilon}{2} sd \log \frac{n}{s} + 2\epsilon ds$$

where $\epsilon = \frac{1}{s} \sum_{v \in S} \epsilon_v$

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Two cases: $s<\alpha\Delta$ and $\alpha\Delta\leq s\leq\frac{n}{2}$...

Compressing Accepted Connections II

Goal: bound $d\sum_{v\in S}(1-\epsilon_v)\log\frac{1}{1-\delta_v}$

Case $s < \alpha\Delta$

Use $\Delta(1-\delta_v) \le s$ and $(\frac{\Delta}{s})^2 > \frac{\Delta}{s}\frac{1}{\alpha} = \frac{\Delta}{s}\frac{n}{\Delta} = \frac{n}{s}$

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together with hypothesis on s and λ, it implies

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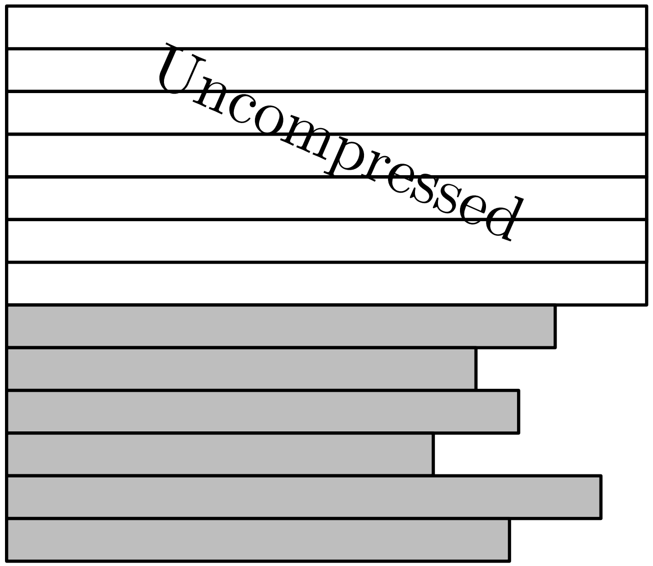
Nodes in $V - S$

Nodes in S



Compressing Rejected Requests (Idea)

With $\ell_v - d'$ bits we encode which requests are rejected.

The hard part is compressing their *destinations*, for which we use the following notions:

Semi-saturated nodes ss_t: accepted connections until time $t-1+$ requests from $V-S$ are $> \frac{dc}{2}$

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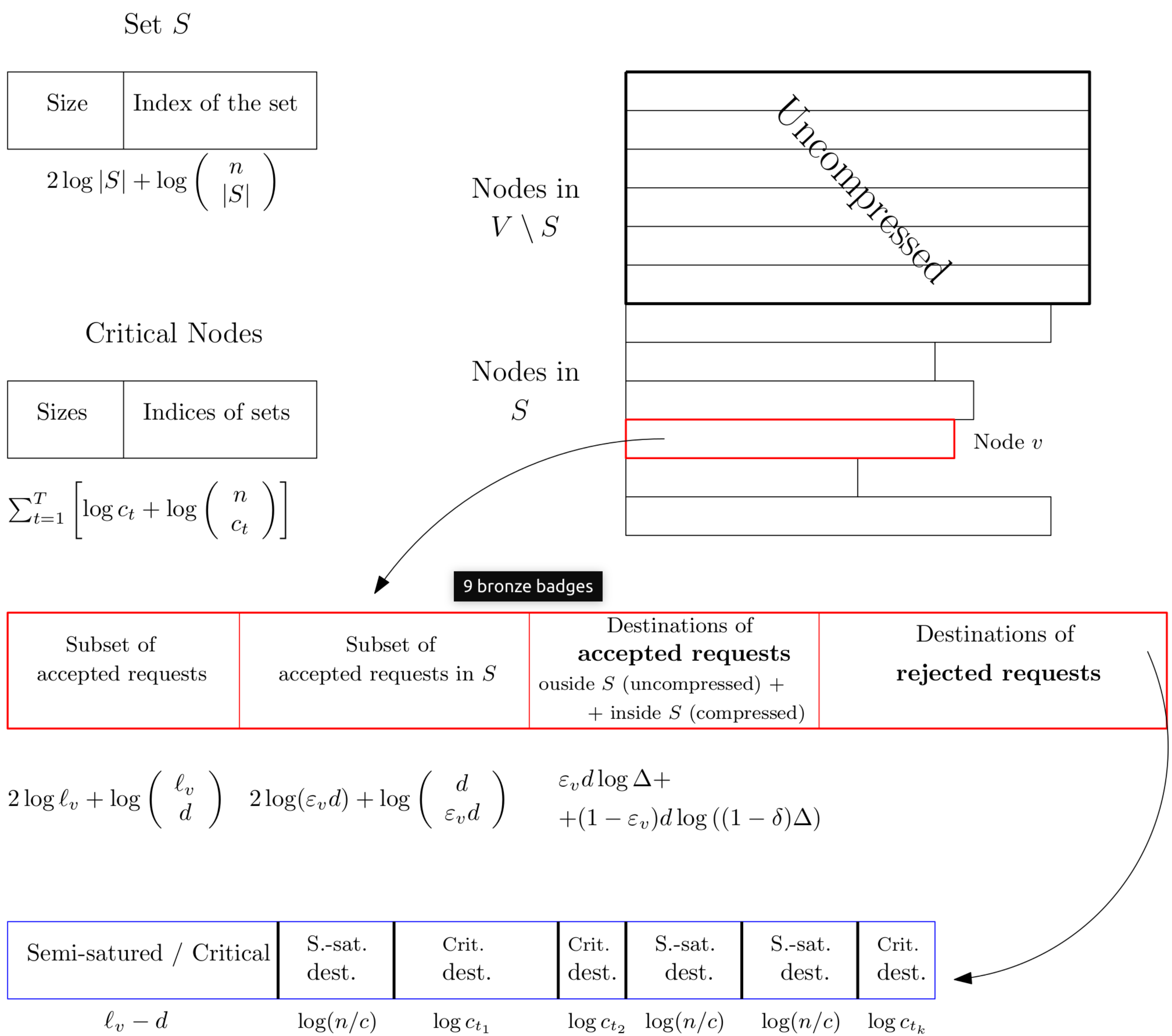
We can then write

$$ss(v) \log \frac{2n}{c} + \sum_1^T rc_t(v) \log c_t$$

Where $rss(v)$ is the number of rejected connections from v to semisaturated nodes and $rc_t(v)$ is the number of rejected connections from v to critical nodes at time t

Compression Summary



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- Investigate robustness of RAES when nodes join or leave the network.

Thank You!