Pooling or Sampling: Collective Dynamics for Electrical Flow Estimation

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Electrical Flow in CS and Biology

Computation of currents and voltages in resistive electrical network is a crucial primitive in many optimization algorithms

- Maximum flow
  - Christiano, Kelner, Madry, Spielman and Teng, STOC'11
  - Lee, Rao and Srivastava, STOC'13
- Network sparsification
  - Spielman and Srivastava, SIAM J. of Comp. 2011
- Generating spanning trees
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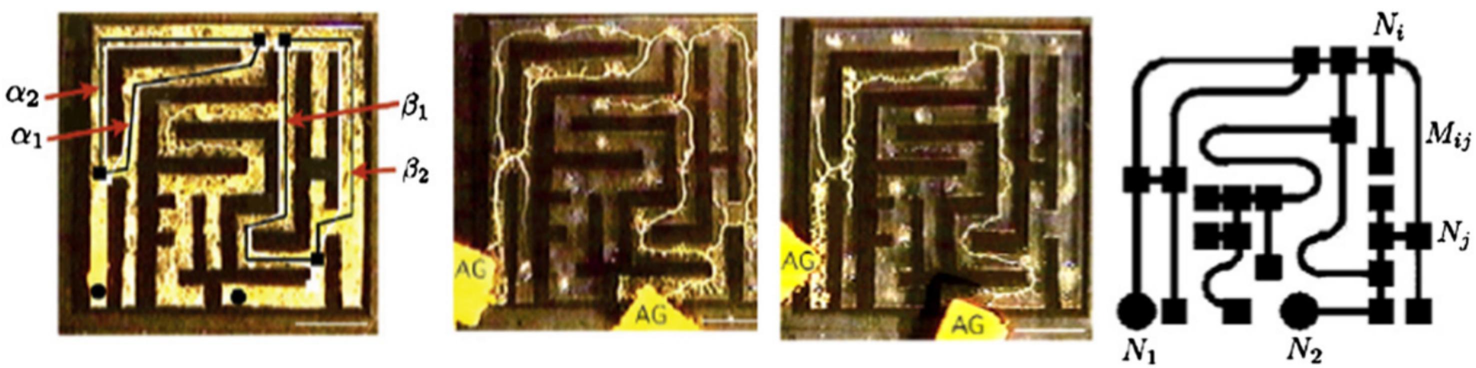
and as model of biological computation

- Physarum polycephalum
- Ants

implicitly solving electrical flow while forming food-transportation networks

Physarum Polycephalum Behavior

- Nakagaki, Yamada and Toth, Nature 2000
- Tero, Kobayashi and Nakagaki J. of Theo. Bio. 2007



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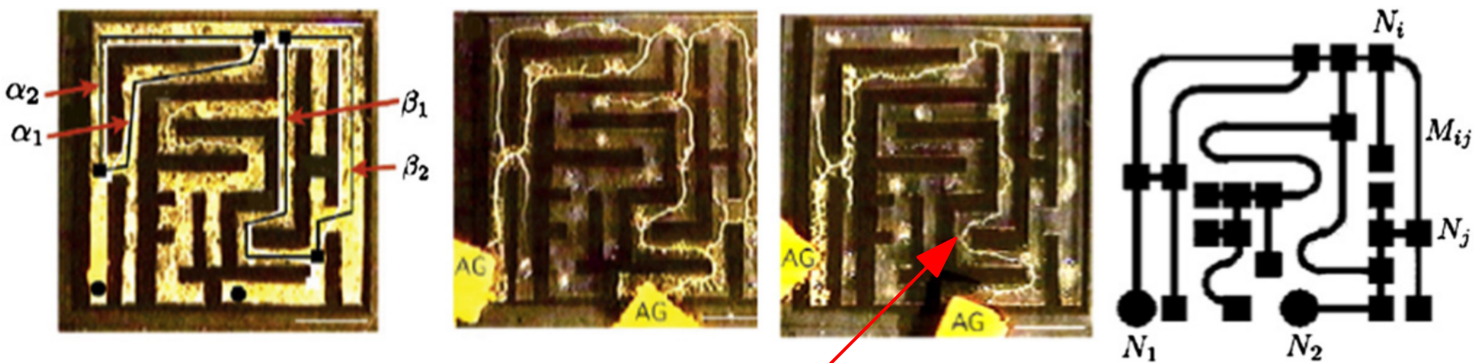
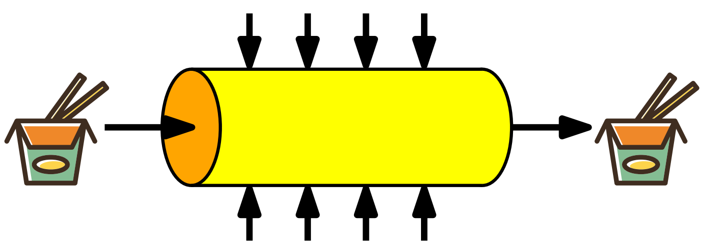
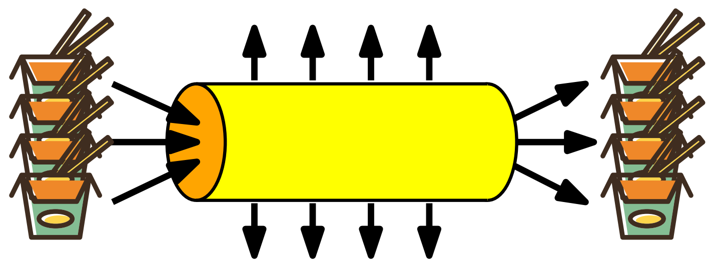


Physarum polycephalum builds *tubes* to transport food.
Amount of food flowing in tube determines growth or deterioration.





Physarum Dynamics as an Algorithm

Dynamics: $\dot{x}_e = |q_e| - x_e$

x_e thickness of segment
q_e amount of nutrient flowing

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$$x(t+1) - x(t) = h(|q(t)| - x(t))$$

Becchetti, Bonifaci, Dirnberger, Karrenbauer and Mehlhorn
ICALP'13: Discretized physarum computes $(1+\epsilon)$-apx.
in $\mathcal{O}(mL(\log n + \log L)/\epsilon^3)$ (m num. of edges, L max. length)

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Many sequels in TCS:
Bonifaci, Mehlhorn and Varma SODA'12, Bonifaci IPL'13, Straszak and Vishnoi ITCS'16, Straszak and Vishnoi SODA'16, Becker et al. ESA'17, ...

How to Compute with Electrical Networks

Physarum have to compute the *pressures*,
i.e. solve Kirchhoff's equations on nodes:

$$Lp = b$$

- edge weights x_e/ℓ_e (ℓ_e length of segment)
- A weighted adjacency matrix
- D diagonal matrix of nodes' volumes
- $L = D - A$ non-normalized graph laplacian
- b is $+1$ on source, -1 on sink, 0 elsewhere

Previous approaches: centralized computation

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Previous approaches: centralized computation

Biologically, computation is achieved through a "microscopic" *local* process: what is it?

Randomized Token Diffusion Process

Process [Ma, Johansson, Tero, Nakagaki and Sumpter, '13]

- At the beginning of each step, K new tokens *appear* at the source
- Each token independently performs a *weighted random walk* at each step
- Each token that hits the sink *disappears*

Estimator

$V_K^{(t)} = \frac{Z_K^{(t)}(u)}{K \cdot \mathsf{vol}(u)}$ where $Z_K^{(t)}(u)$ *number of tokens on* u

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Cfr. Doyle and Snell, '84 & Tetali, '91:
Times random walk *transits on an edge before hitting sink*

Analysis of Random Process

Theorem. In expectation the process converges to a *valid potential* with rate

$$1 - \frac{1 - \sqrt{1 - (\theta/\mathrm{vol}_{\mathrm{max}})^2}}{2(n-1)} \sum_i \frac{w_{in}}{w_{in} + \mathrm{vol}_{\mathrm{max}} - \sqrt{\mathrm{vol}_{\mathrm{max}}^2 - \theta^2}}$$

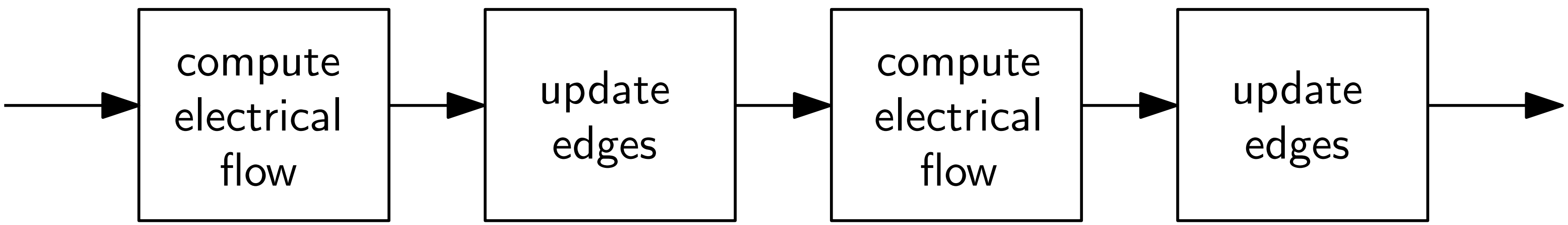
where θ is the edge expansion of the graph with sink removed.

Proof.

- Bound w.r.t. spectral radius of P with row and column of sink set to zero, then
- relate bound to eigenvalue of non-norm. laplacian of graph with sink removed, finally
- use known relation to edge expansion.

Open Problem: Analysis of *Distributed* Physarum?

Physarum dynamics et sim.:
Compute electrical flow, then *update edge-weigths*



$Lp = b$ $\dot{x}_e = |q_e| - x_e$ $L'p = b$ $\dot{x}_e = |q_e| - x_e$

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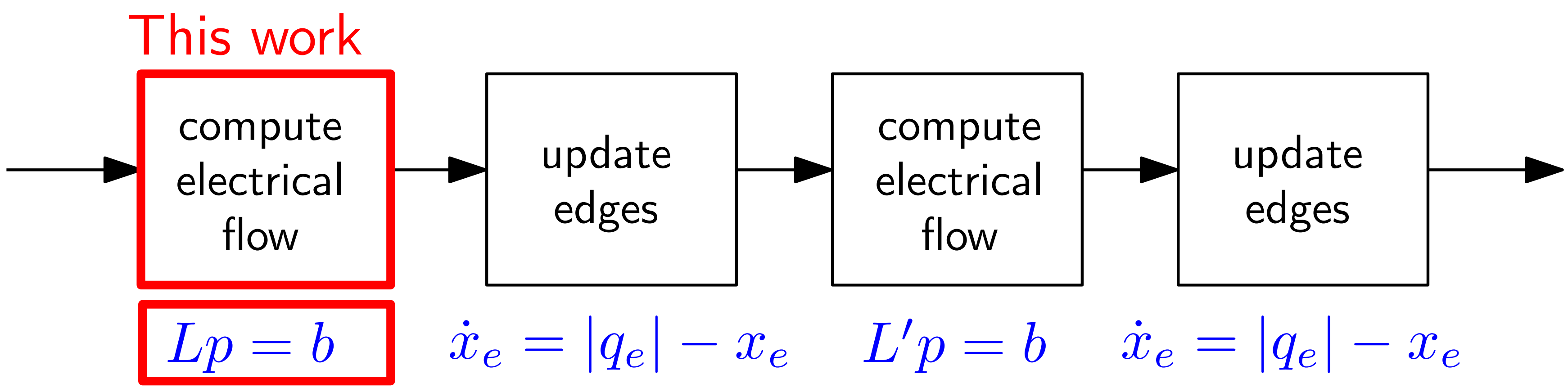

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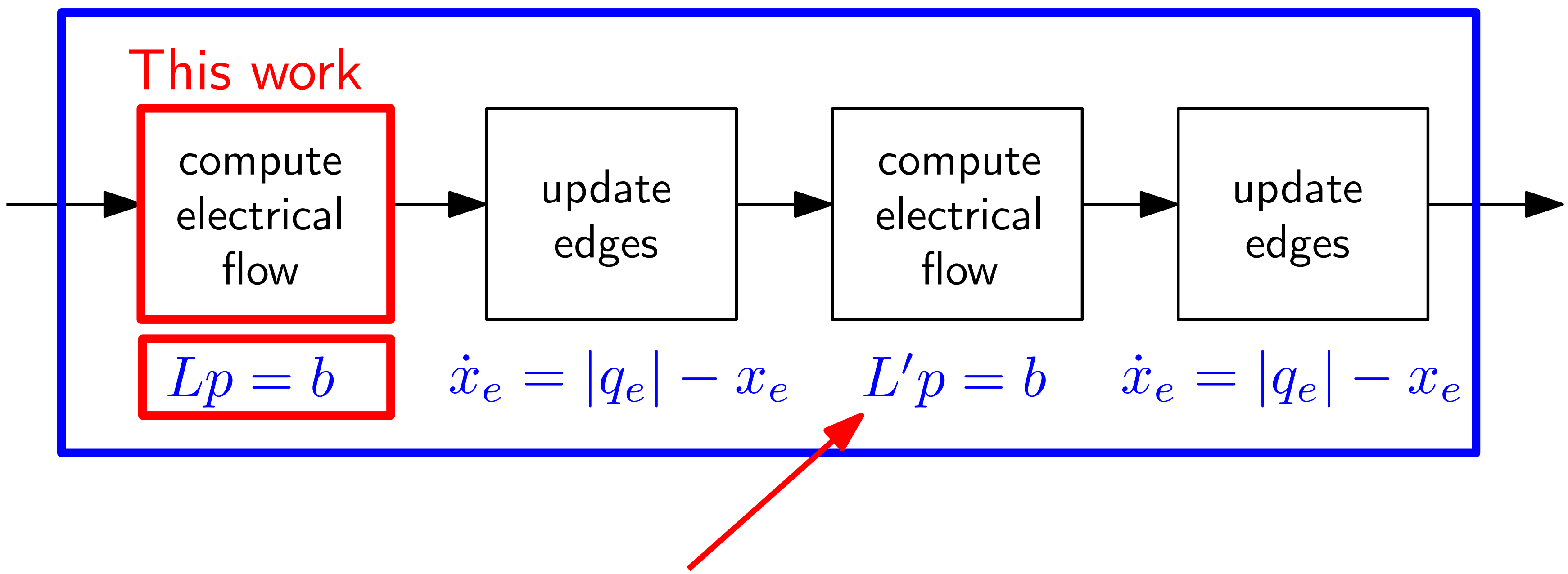

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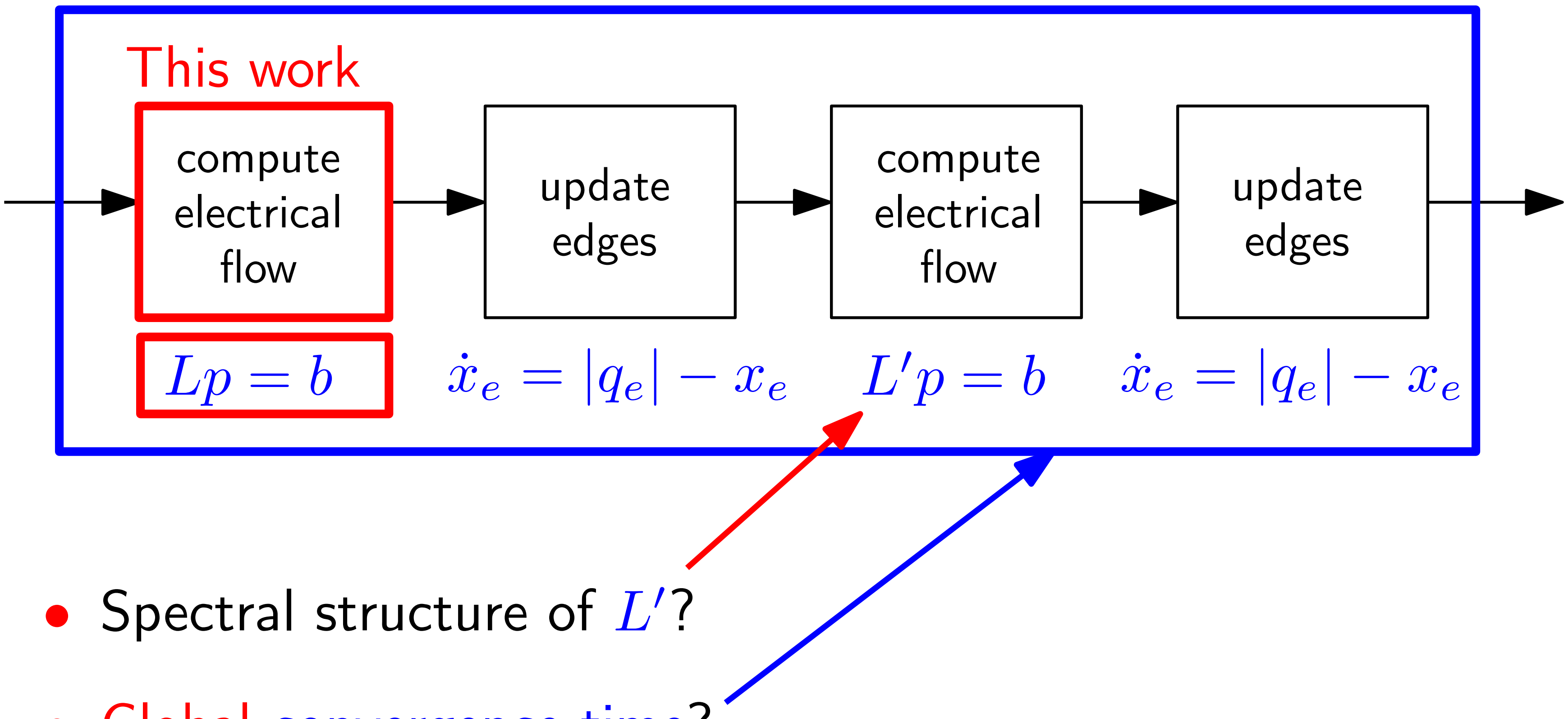


- Spectral structure of L'?
- Global convergence time?

Thank You!

Distributed Jacobi's Method

Thm. Let

$$\tilde{p}(t+1) = P\tilde{p}(t) + D^{-1}b$$

where $P = D^{-1}A$ is the transition matrix of graph. The system converges to a *valid potential* with rate

$$\|e_{\perp}(t)\| \leq \sqrt{\frac{\text{vol}_{\max}}{\text{vol}_{\min}}}\,\sqrt{2\phi}^{\,t}\,\|e_{\perp}(0)\|$$

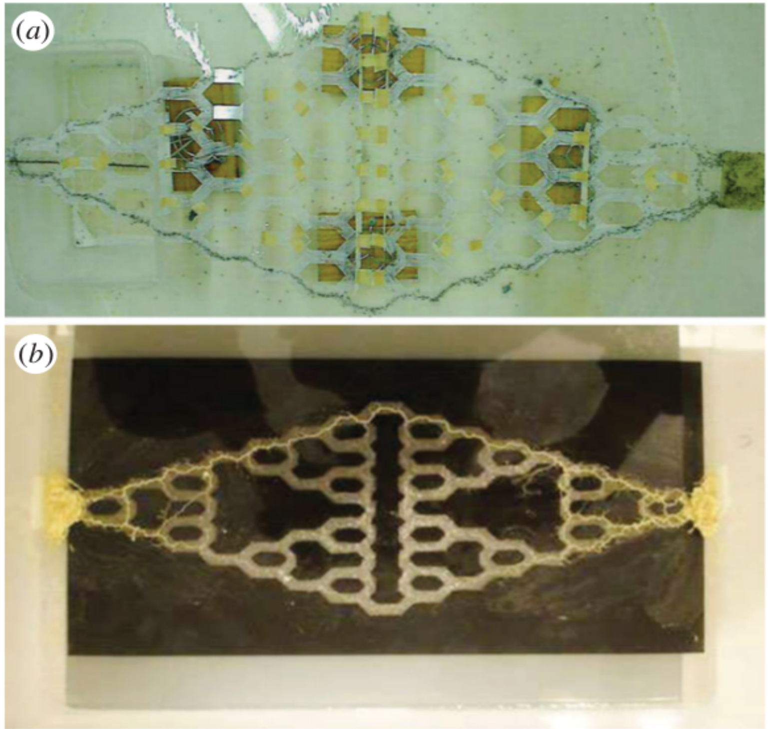
where $e_{\perp}(t)$ is the component of the error orthogonal to $\mathbf{1}$ and ϕ is the *graph conductance*.

Proof. New analysis of Jacobi's iterative method exploiting structure of laplacian.

The Slime Mold *Physarum Polycephalum*

electric network	***Physarum***	**ant trails**
length in space	length in space	length in space
potential/voltage	amount of nutrient	number of ants
current	flow of nutrient	flow of ants
conductivity	thickness of tube	pheromone concentration
capacitance	transport efficiency	total pheromone density
reinforcement intensity	tube expansion rate	pheromone drop rate
conductivity decrease rate	tube decay rate	evaporation rate

Ma, Johansson, Tero, Nakagaki and Sumpter, J. of the Royal Society Interface '13



Message Complexity and Stochastic Accuracy

Lemma. As $t \to \infty$, the *expected message complexity* per round of Token Diffusion Algorithm is $O(K\, n\, \mathrm{vol}_{\max} \cdot E)$, where $E = p^{\intercal} L p$ is the *energy* of the electrical flow.

Lemma. For any K, $0 < \epsilon, \delta < 1$, t and u, such that $p_u^{(t)} \geq \frac{3}{\epsilon^2 K \mathrm{vol}(u)} \ln \frac{2}{\delta}$, the estimator provides an (ϵ, δ)-approximation* of $p_u^{(t)}$.

Vice versa. (ϵ, δ)-approximation of the potentials $p_u^{(t)}$ greater than $p_\star^{(t)}$ is achieved by setting $K \geq \frac{3}{\epsilon^2 p_\star^{(t)} \mathrm{vol}(u)} \ln \frac{2}{\delta}$.

Proofs. Chernoff bound requires $Y > \frac{3 \ln \frac{1}{\delta}}{\epsilon^2}$.

* X gives (ϵ, δ)-approximation of Y if $\mathbf{P}\left(|X - Y| > \epsilon Y\right) \leq \delta$.