Consensus needs *Broadcast* in Noiseless Models but can be Exponentially Easier in the Presence of Noise

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Adfocs

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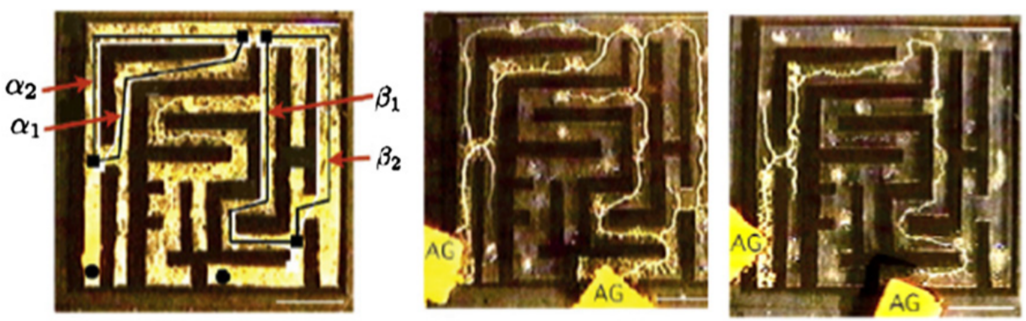
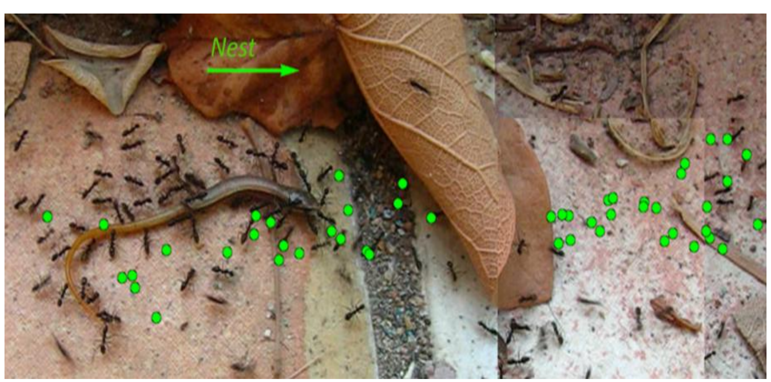
Natural Algorithms

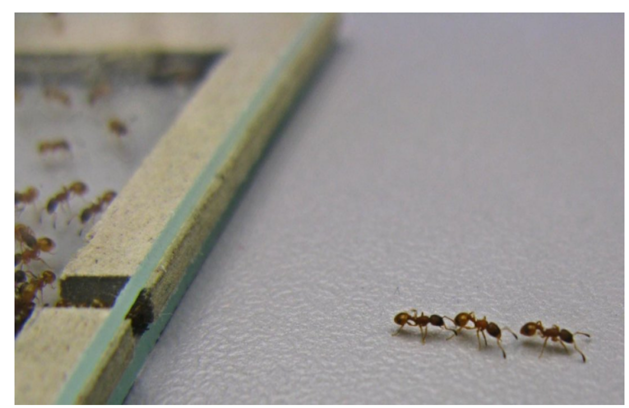
How do flocks of birds synchronize their flight? [Chazelle '09]



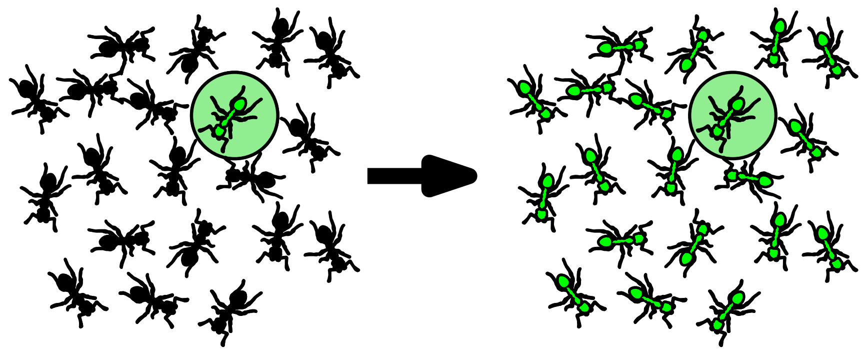
How does Physarum polycephalum finds shortest paths? [Mehlhorn et al. 2012-...]



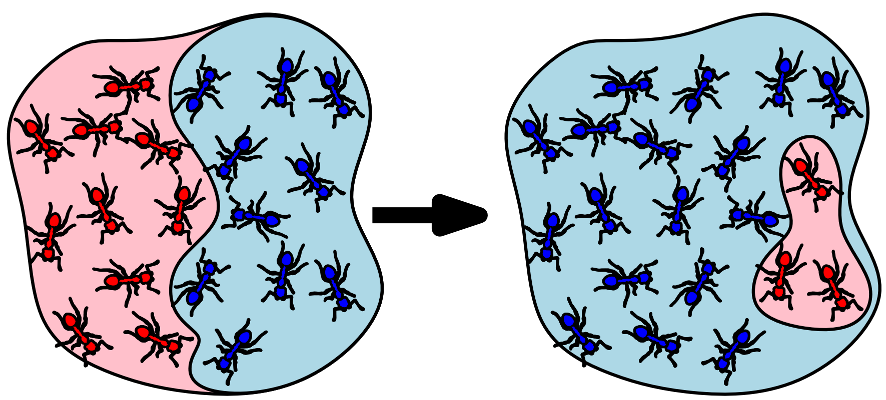
How ants perform collective navigattion? How do they decide where to relocate their nest?



Noisy vs Noiseless Broadcast and Consensus



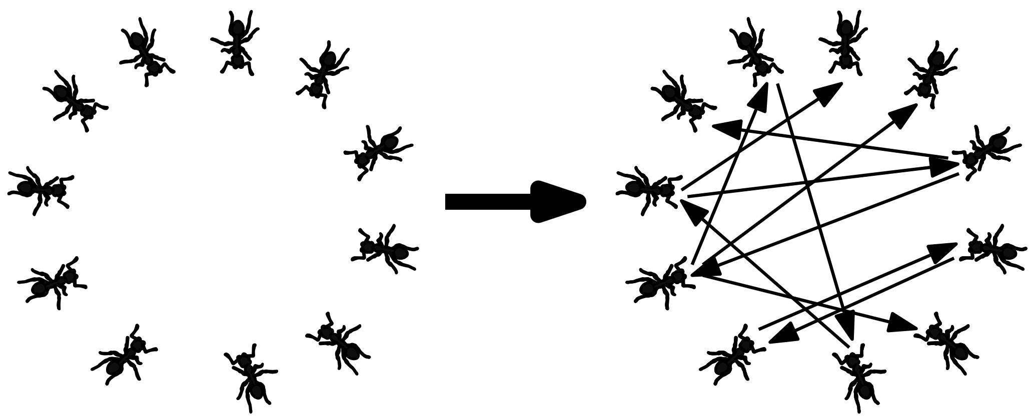
Broadcast. All agents eventually receive the message of the source.



(Valid) δ-Consensus. All agents but a fraction δ, eventually support the value initially supported by one of them.

Noisy & Stochastic Interactions

Stochastic Interactions.
At each round, each agent receives a message from another random agent.



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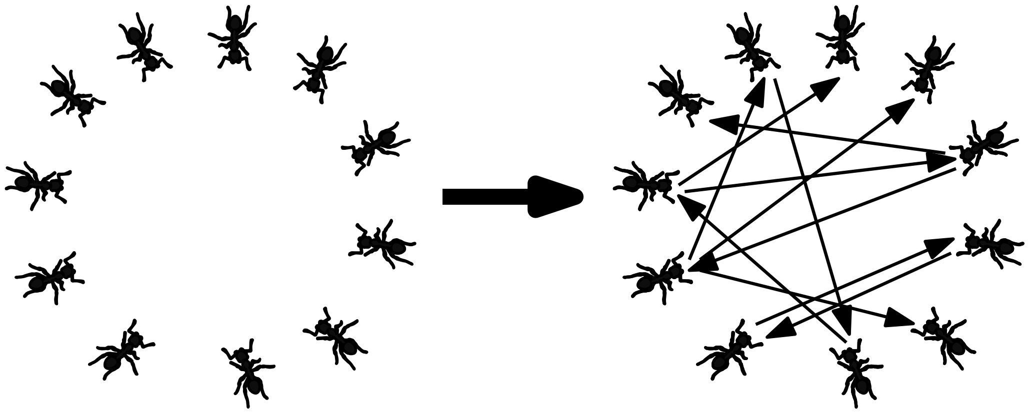
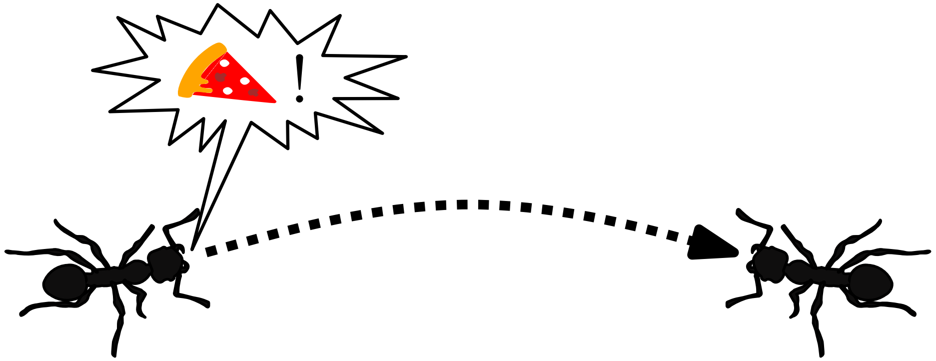


Noisy Communication. Before being received, each bit is flipped with probability $1/2 - \epsilon_n$.



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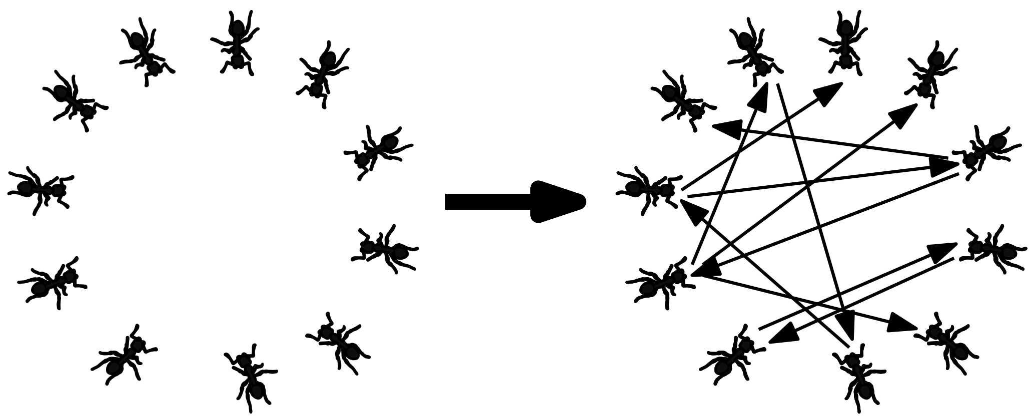

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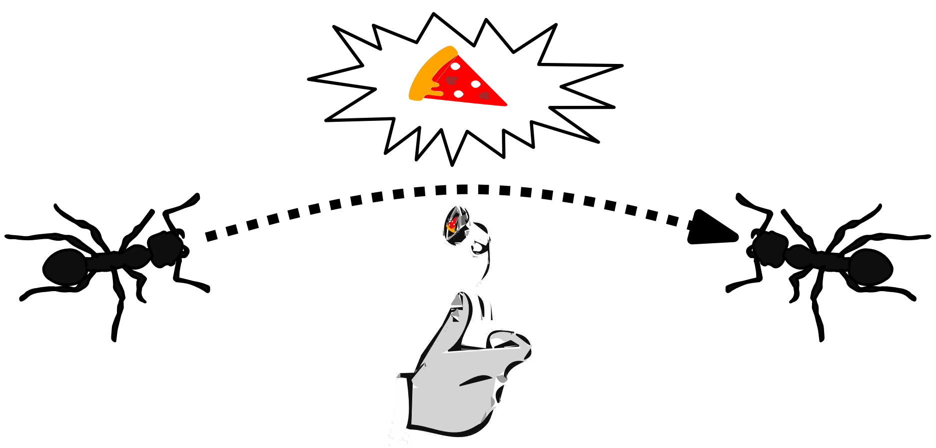

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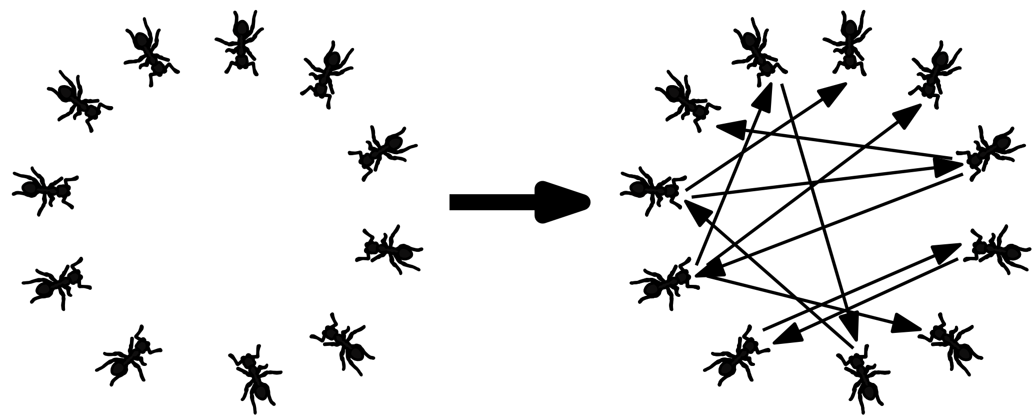

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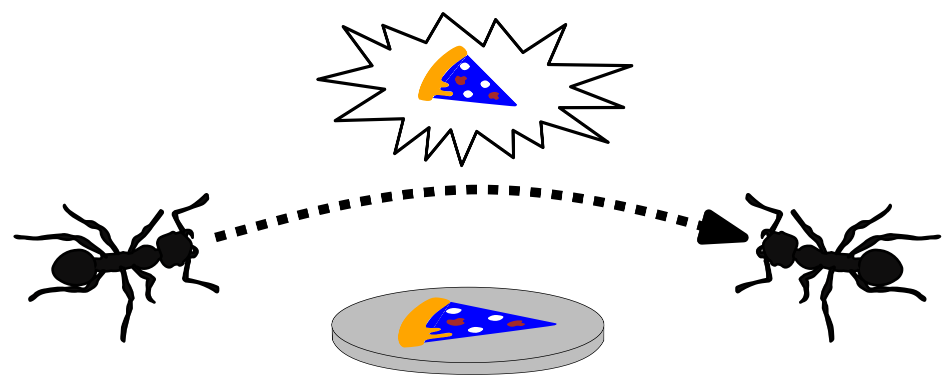

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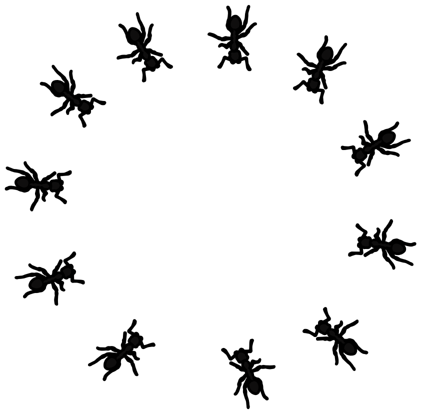

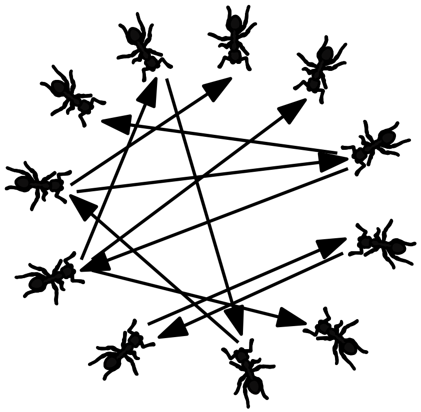

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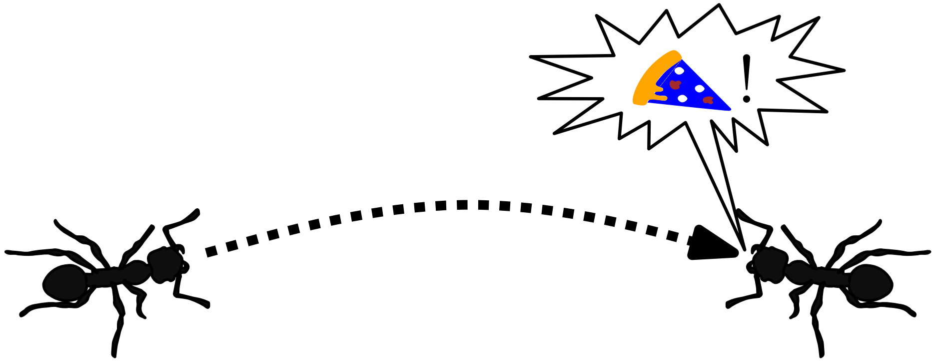
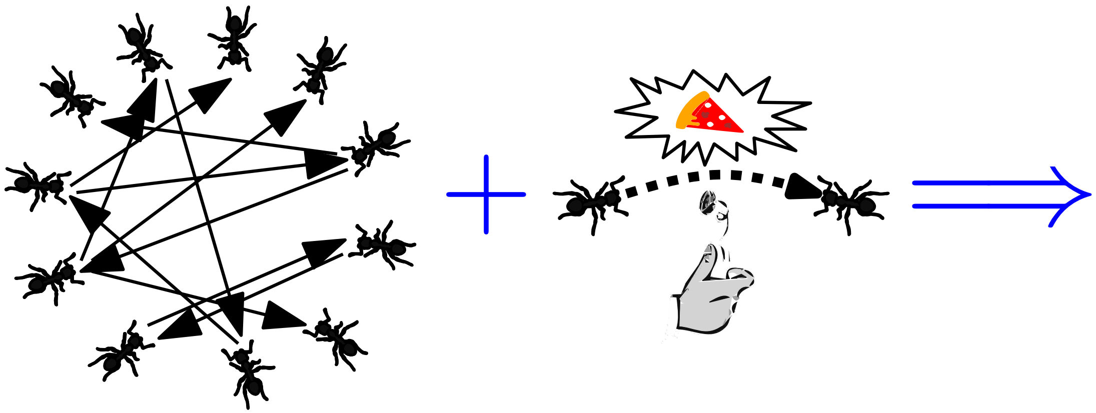


Lower Bounds and Reductions



Noisy Consensus:
$\Theta(\frac{\log n}{\epsilon^2})$ rounds

Noisy Broadcast:
$\Theta(n \cdot \frac{\log n}{\epsilon^2})$ rounds

Noisy Broadcast is *exponentially harder*
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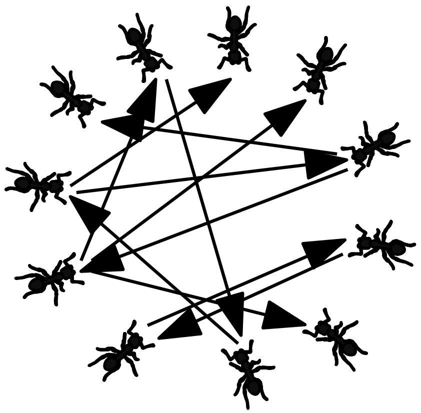

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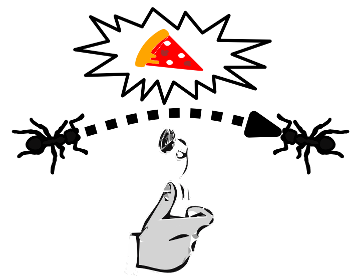

$\Longrightarrow$

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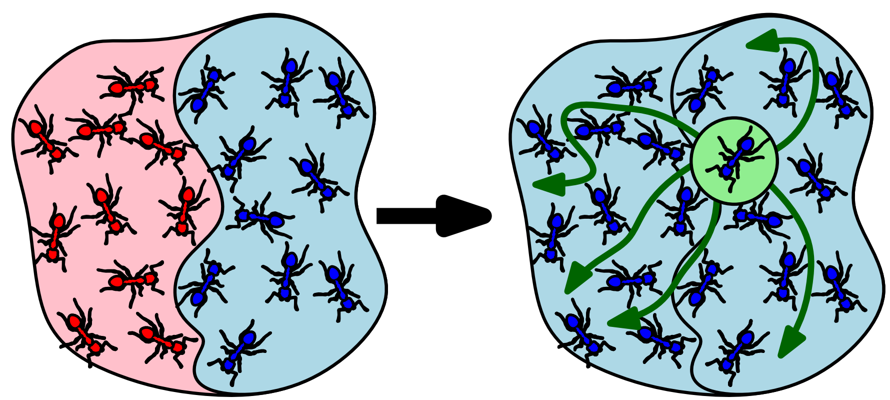
Broadcast $\Longrightarrow$ Consensus

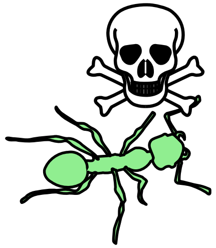
Noiseless Consensus $\Longrightarrow$ **Noiseless** (variant of) Broadcast

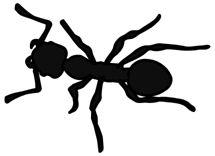


Noiseless Consensus and Broadcast are "*equivalent*"

Consensus $\Longrightarrow$ "Broadcast"

Def. Given agent s, we call an agent infected if it is s or it receives any message from an infected agent. Protocol $\mathcal{P}$ is δ-infective w.r.t. s if *infects* all but a fraction δ of agents.





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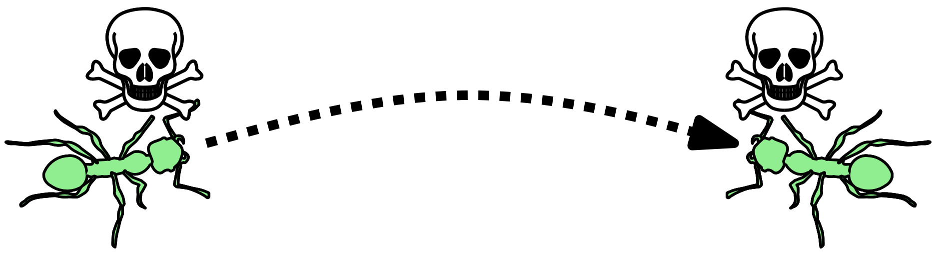

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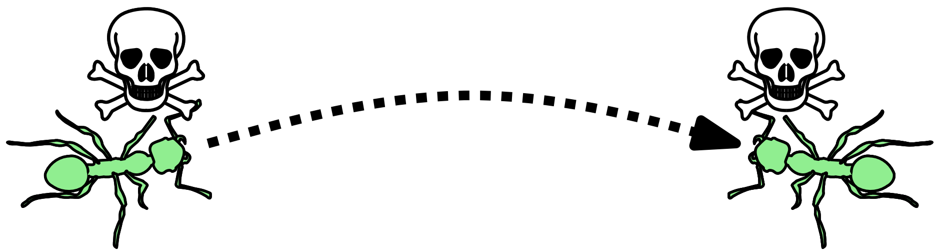

Thm. Let $\mathcal{P}$ be a δ-consensus protocol with probability $1 - o(1/n)$. There is an agent s and initial inputs to agents such that $\mathcal{P}$ is $(1 - 2\delta)$-infective with probability $\geq 1/(2n)$.

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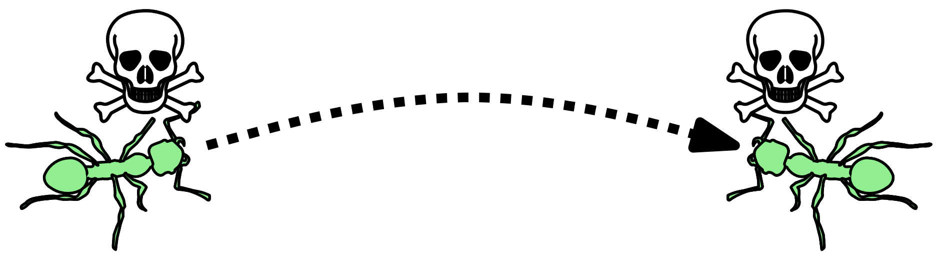

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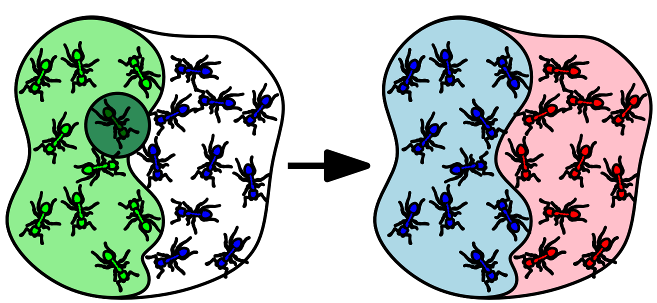
Corollary. Let $\mathcal{T}$ be a resource of the distributed system S. If no protocol can infect more than $(1 - 2\delta)$ fraction of agents with high probability, w.r.t. any source, without exceeding t_b units of $\mathcal{T}$, then any δ-consensus protocol with high probability must exceeed t_b.



Proof in 9 Steps

1. Label nodes $v_1, ..., v_n$. x_k is initial configuration with $v_1, ..., v_k$ having input 0, while others have input 1.

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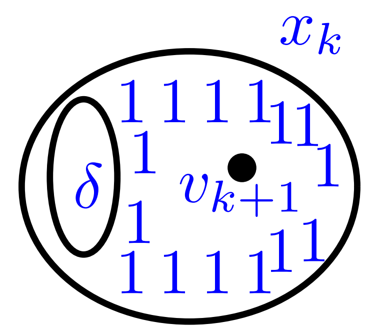

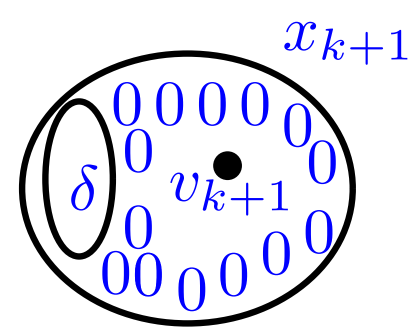

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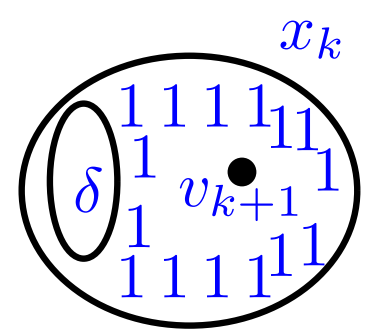

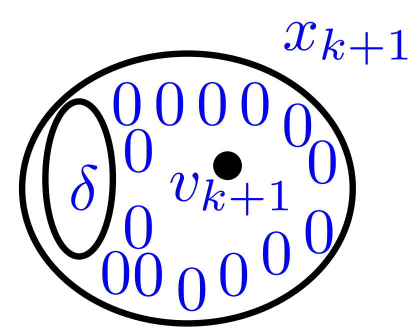

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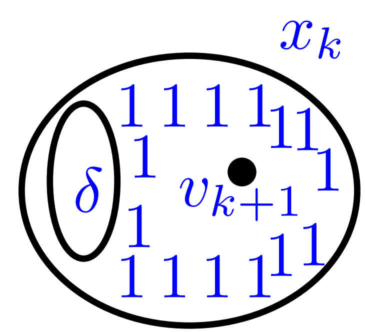

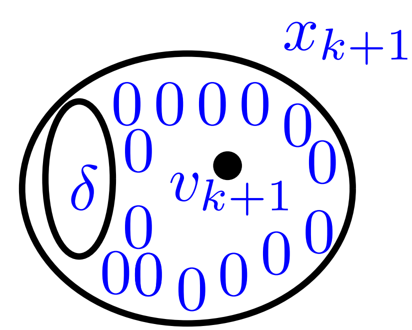

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9. $\leq P(\neg \mathcal{S} \vee |I_{k+1}| > (1-2\delta)n) \leq o(1/n) + P(|I_{k+1}| > (1-2\delta)n)$ $\square$

Thank You!