Pooling or Sampling: Collective Dynamics for Electrical Flow Estimation

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joint work with L. Becchetti[2] and V. Bonifaci[3]

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Electrical Networks for Optimization

Computation of currents and voltages in resistive electrical network is a crucial primitive in many optimization algorithms

- Maximum flow
  - Christiano, Kelner, Madry, Spielman and Teng, STOC'11
  - Lee, Rao and Srivastava, STOC'13
- Network sparsification
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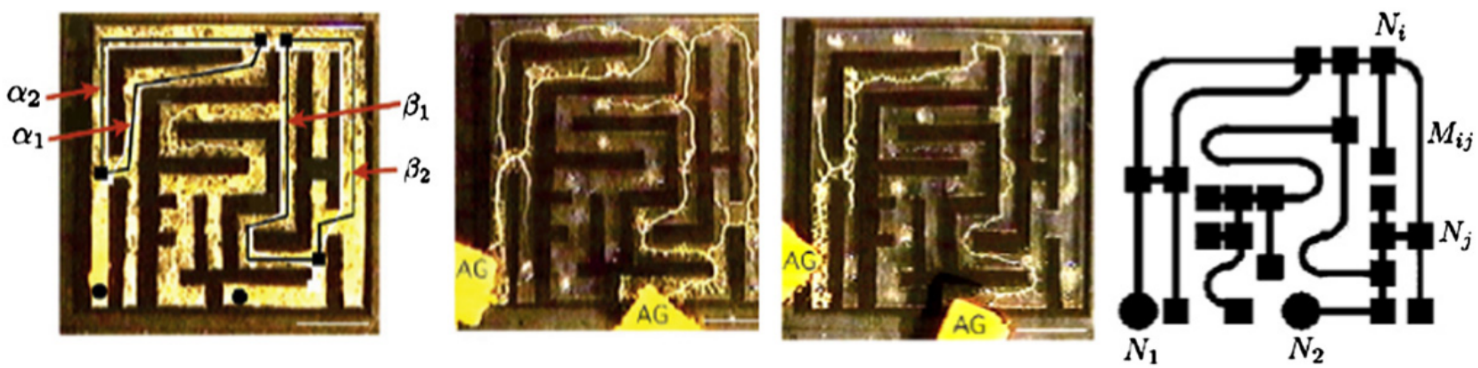
and as model of biological computation

- Physarum polycephalum
- Ants

implicitly solving electrical flow while forming food-transportation networks

Physarum Polycephalum Behavior

- Nakagaki, Yamada and Toth, Nature 2000
- Tero, Kobayashi and Nakagaki J. of Theo. Bio. 2007



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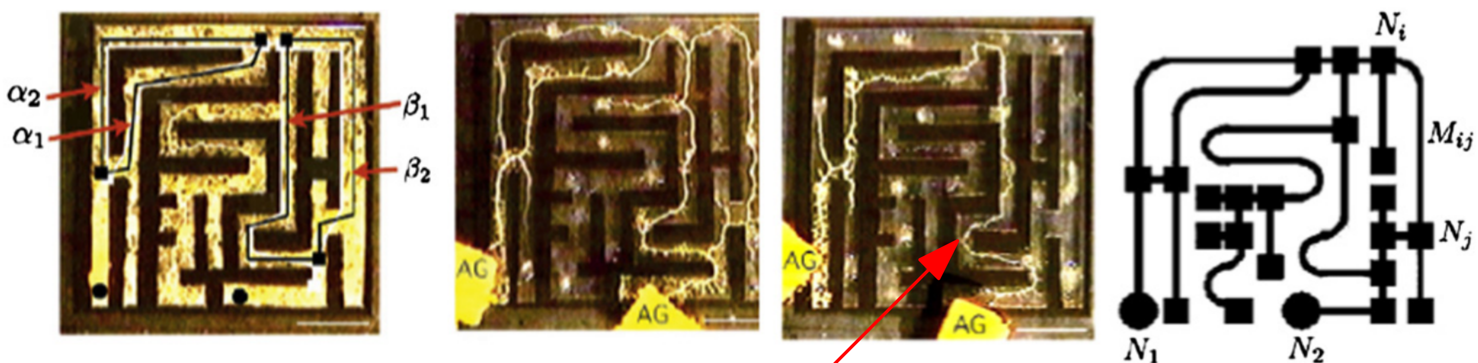
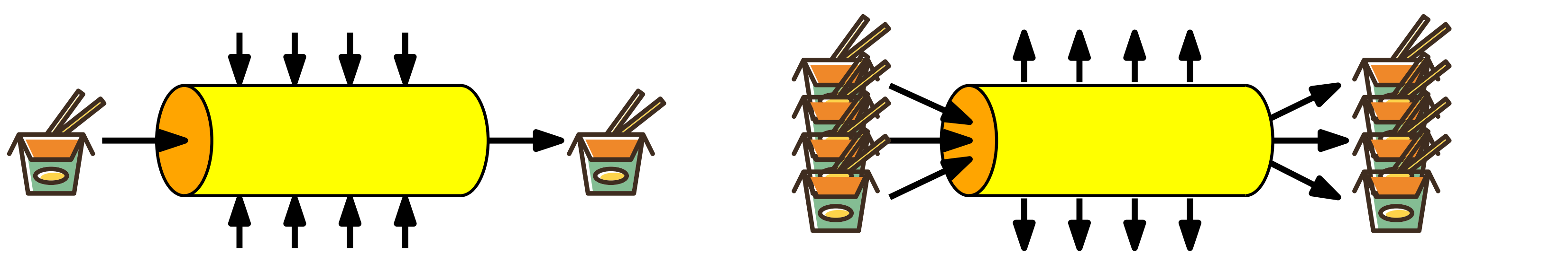


Physarum polycephalum builds *tubes* to transport food.
Amount of food flowing in tube determines growth or deterioration.



Physarum Polycephalum Dynamics

For each edge e and node u

- ℓ_e length
- x_e thickness (conductivity)
- q_e food flow (current)
- $r_e = \ell_e / x_e$ resistance to flow

$\boxed{\text{Dynamics: } \dot{x}_e = |q_e| - x_e}$

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- Flows relates to pressures by
  $q_{(u,v)} = (p_u - p_v)/r_e$
  (Ohm's law)
- there are pressures $p(u)$:
  $\forall$cycle $u_1, \ldots, u_\ell$,
  $\sum_i (p(u_{i+1}) - p(u_i)) = 0$
  (Kirchhoff potential law)

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- flow conservation: $\sum_{v \sim u} q_{(u,v)} = b(u)$ (Kirchhoff current law)
- there are demands $b(u)$:
  - 1 on source,
  - -1 on sink,
  - 0 o/w

Physarum Dynamics as an Algorithm

Bonifaci, Mehlhorn and Varma SODA'12:
Physarum dynamics converges on all graphs
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Euler's discretization

$$x(t+1) - x(t) = h(|q(t)| - x(t))$$

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Many sequels in TCS:
Bonifaci IPL'13, Straszak and Vishnoi ITCS'16, Straszak and Vishnoi SODA'16, Becker et al. ESA'17, ...

How to Compute with Electrical Networks

Physarum have to solve Kirchhoff's equations

$\sum_{v \sim u} q_{(u,v)} = \sum_{v \sim u} (p_u - p_v)/r_e = b(u)$

or $\boxed{Lp = b}$

- edge's weight x_e/ℓ_e
- D diagonal matrix of nodes' volumes
- A weighted adjacency matrix
- $L = D - A$

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Previous approaches: centralized computation

- Can be accomplished if every node is agent that follows elementary protocol?
(*biologically*: what happens microscopically?)
- If yes, what is convergence time and communication overhead?

Distributed Jacobi's Method

Jacobi's iterative method (Varga, 2009):

Bound on convergence rate w.r.t. *graph conductance*
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$$Lp = (D - A)p = b \implies p = \underbrace{D^{-1}A}_{P}p + D^{-1}b$$

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Error $e(t) = p - \tilde{p}(t) = e_{\perp}(t) + \alpha\mathbf{1}$
(p *doesn't care about* α: $L\mathbf{1} = 0$!)

Analysis of Deterministic Process

The new error is $e_\perp(t+1) = Pe_\perp(t) - (\alpha(t+1) - \alpha(t))\mathbf{1}$
thus $e_\perp(t) = \left(I - \frac{1}{n}\mathbf{1}\mathbf{1}^\mathsf{T}\right) P^t e_\perp(0)$.

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Thus

$$P^t = (D^{-1}A)^t = (D^{-\frac{1}{2}}ND^{\frac{1}{2}})^t = D^{-\frac{1}{2}}N^tD^{\frac{1}{2}}.$$

Observe that

- N has n orthonormal eigenvec. $\vec{x}_1, \ldots, \vec{x}_n$, corresponding to eigenvectors $\vec{y}_1, \ldots, \vec{y}_n$ of P via $\vec{x}_i = D^{1/2}\vec{y}_i$ for each i.
- Both $\vec{x}_i$ and $\vec{y}_i$, for each i, are associated to the same eigenvalue ρ_i of P.

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$$\|e_\perp(t)\| \leq \sqrt{\frac{\mathrm{vol}_{\max}}{\mathrm{vol}_{\min}}} \max(|\rho_2|, |\rho_n|)^t \|e_\perp(0)\|$$

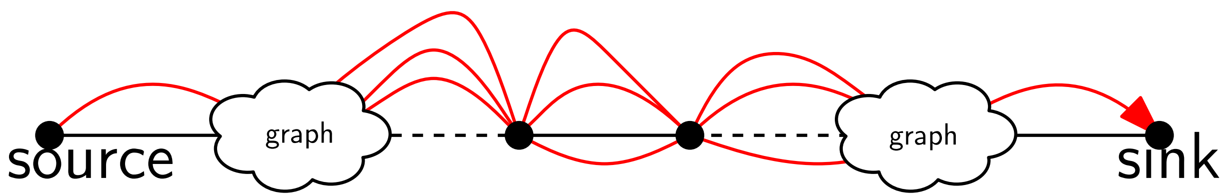
Conductance by Cheeger's inequality

Randomized Token Diffusion Process

Doyle and Snell, '84 & Tetali, '91:

Times a random walk transits through given edge *until hitting* the sink

- *global* requirement
- no accuracy and msg. complexity bounds



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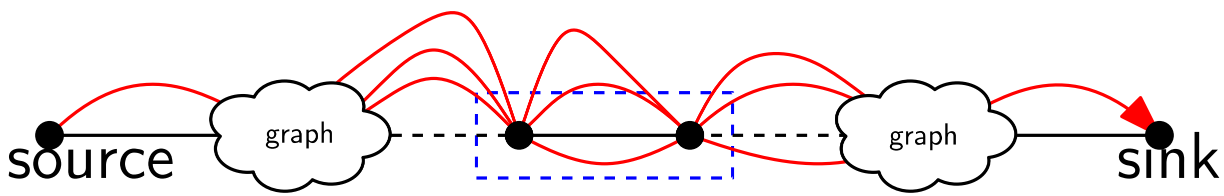

Our's:

How many tokens are on a node

- *local* requirement
- accuracy and msg. complexity w.r.t. *edge expansion*

Randomized Token Diffusion Process

Process

- At the beginning of each step, K new tokens *appear* at the source
- Each token independently performs a *weighted random walk* at each step
- Each token that hits the sink *disappears*

Estimator

$V_K^{(t)} = \frac{Z_K^{(t)}(u)}{K \cdot \mathsf{vol}(u)}$ where $Z_K^{(t)}(u)$ number of tokens on u

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Lemma. Let $\mathbb{E}[V_K^{(t)}(u)] = (\mathbf{p}^{(t)})_u$, then

$$\begin{cases} \mathbf{p}^{(0)} & = \vec{0}, \\ \mathbf{p}^{(t+1)} & = \underline{P}\,\mathbf{p}^{(t)} + D^{-1}\underline{b}, \end{cases}$$

with $\underline{P}$ and $\underline{b}$ obtained by zeroing out entries on row and column of sink.

Analysis of Random Process

Lemma. The spectral radius of $\underline{P}$, $\underline{\rho}$, satisfies $\underline{\rho} = 1 - \sum_{i=1}^{n} v_i \cdot P_{i,\text{sink}} / \|v_1\|$, where $\vec{v}_1$ is left Perron eigenvector of $\underline{P}$.

Theorem. System above converges to a *valid potential* with rate $\underline{\rho}$.

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Theorem. $1 - \underline{\rho} \geq \frac{\overline{\lambda}_2}{2\text{vol}_{\max}(n-1)} \sum_i \frac{w_{in}}{w_{in} + \overline{\lambda}_2}$
where $\overline{\lambda}_2$ is 2nd smallest eigenvalue of non-normalized laplacian of graph with sink removed.

Connecting with *edge expansion*:
it is known
$\lambda_2(\mathcal{G}) \geq \text{vol}_{\max} - (\text{vol}_{\max}^2 - \theta(\mathcal{G})^2)^{1/2}$.

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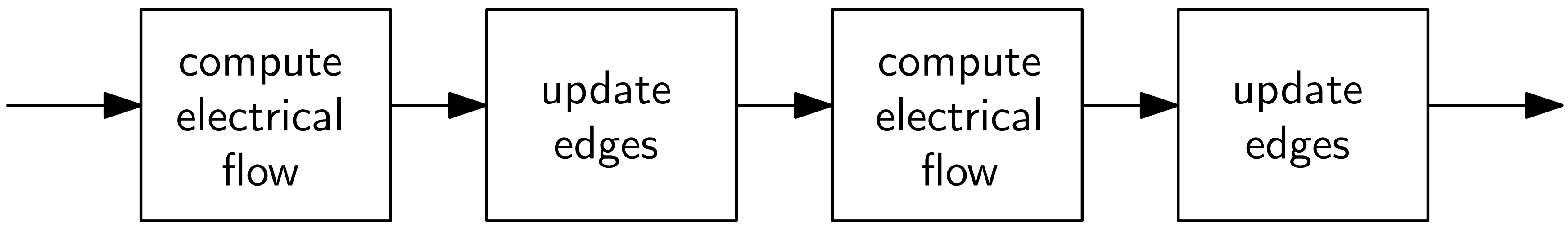
Remark. As $t \to \infty$, the *expected message complexity* per round of Token Diffusion Algorithm is $O(K\, n\, \text{vol}_{\max} \cdot E)$, where $E = p^\intercal L p$ is the *energy* of the electrical flow.

Open Problem: Analysis of *Distributed* Physarum?

Physarum dynamics et sim.:
Compute electrical flow, then *update edge-weigths*



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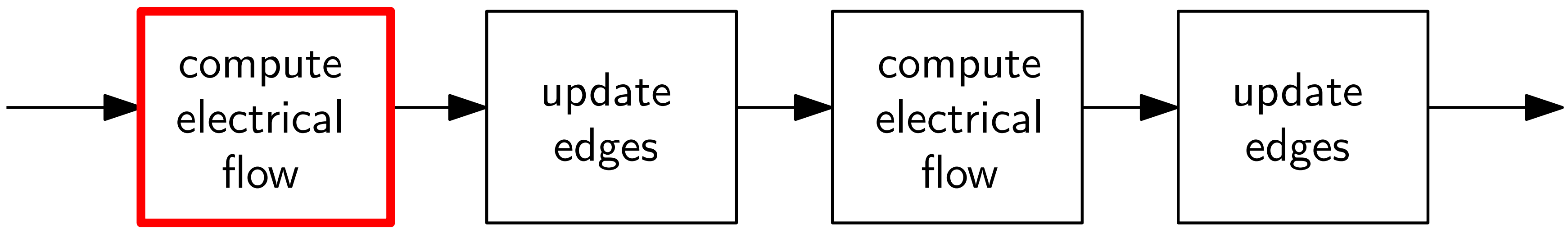
This work



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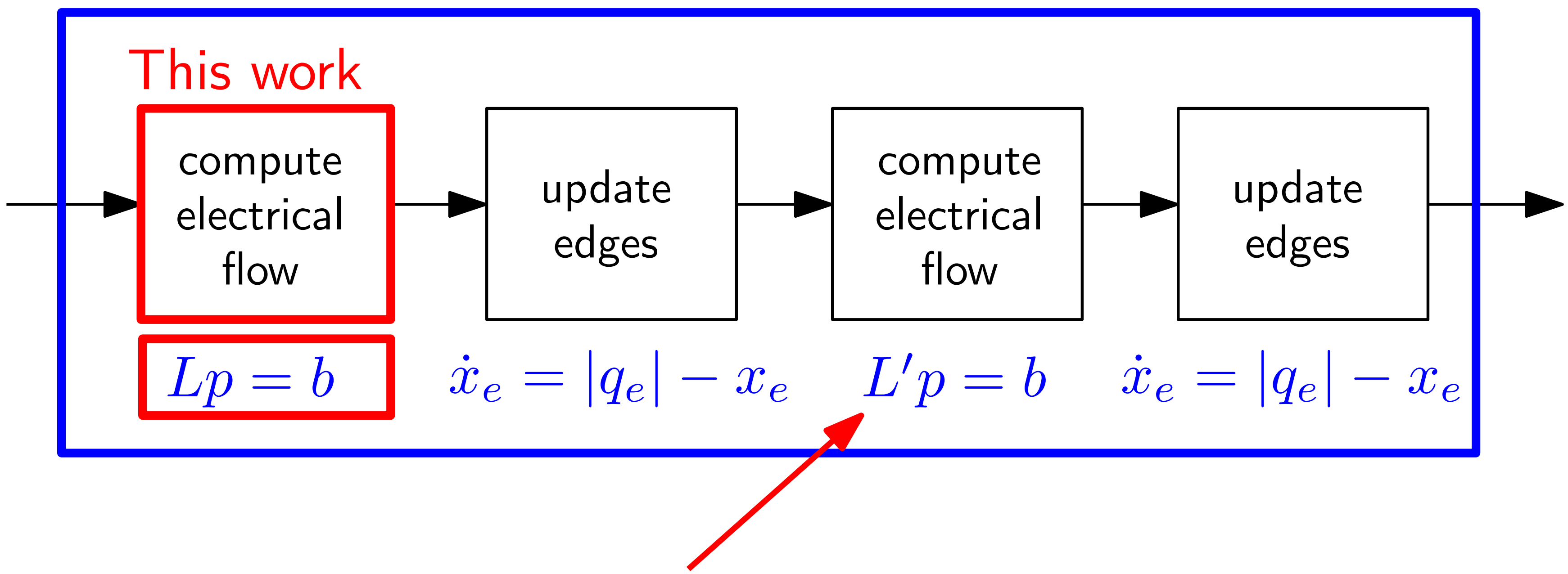

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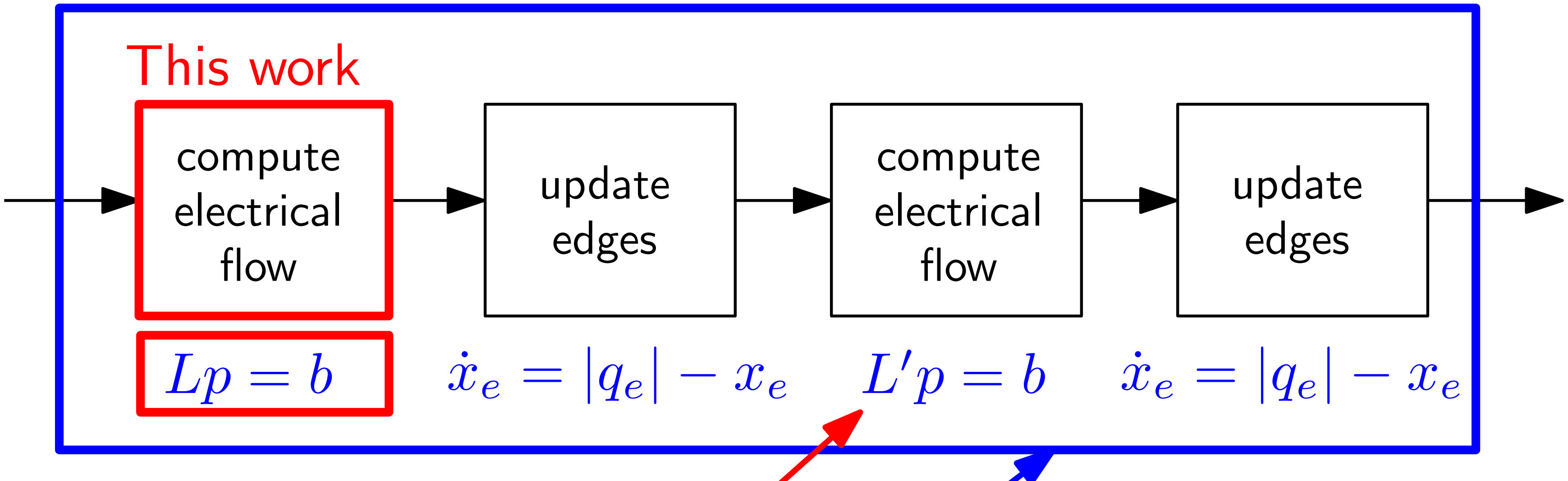

- Spectral structure of L'?
- Global convergence time?

Thank You!

Stochastic Accuracy

X gives (ϵ, δ)-approximation of Y if $\mathbf{P}\left(|X - Y| > \epsilon Y\right) \leq \delta$.

Lemma. For any K, $0 < \epsilon, \delta < 1$, t and u, such that $p_u^{(t)} \geq \frac{3}{\epsilon^2 K \mathrm{vol}(u)} \ln \frac{2}{\delta}$, the estimator provides an (ϵ, δ)-approximation of $p_u^{(t)}$.

Vice versa. (ϵ, δ)-approximation of the potentials $p_u^{(t)}$ greater than $p_\star^{(t)}$ is achieved by setting $K \geq \frac{3}{\epsilon^2 p_\star^{(t)} \mathrm{vol}(u)} \ln \frac{2}{\delta}$.

Proofs. Chernoff bound requires $Y > \frac{3 \ln \frac{1}{\delta}}{\epsilon^2}$.

The Slime Mold *Physarum Polycephalum*

electric network	*Physarum*	ant trails
length in space	length in space	length in space
potential/voltage	amount of nutrient	number of ants
current	flow of nutrient	flow of ants
conductivity	thickness of tube	pheromone concentration
capacitance	transport efficiency	total pheromone density
reinforcement intensity	tube expansion rate	pheromone drop rate
conductivity decrease rate	tube decay rate	evaporation rate

Ma, Johansson, Tero, Nakagaki and Sumpter, J. of the Royal Society Interface '13