Friend or Foe? Population Protocols can perform Community Detection

Emanuele Natale[◇]

joint work with

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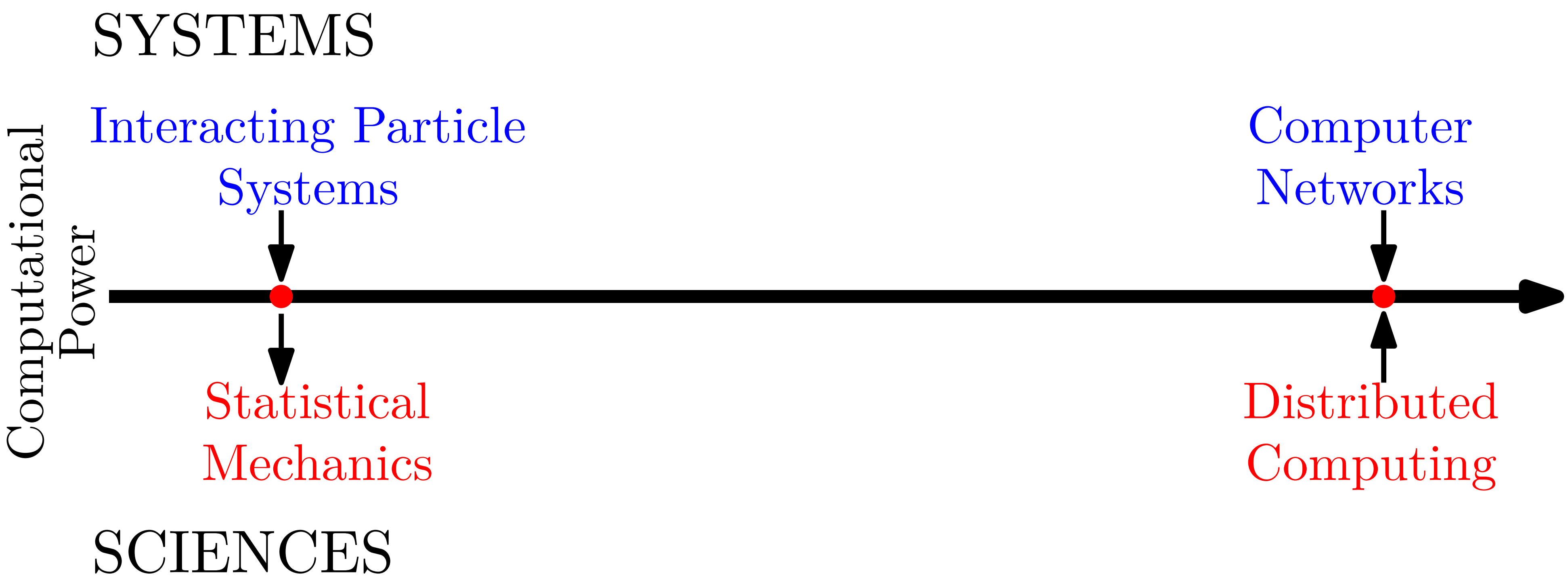
INSTITUT DE RECHERCHE EN INFORMATIQUE FONDAMENTALE

IRIF Algorithms and Complexity seminar
21 March 2017, Paris

Communication in *Simple* Systems



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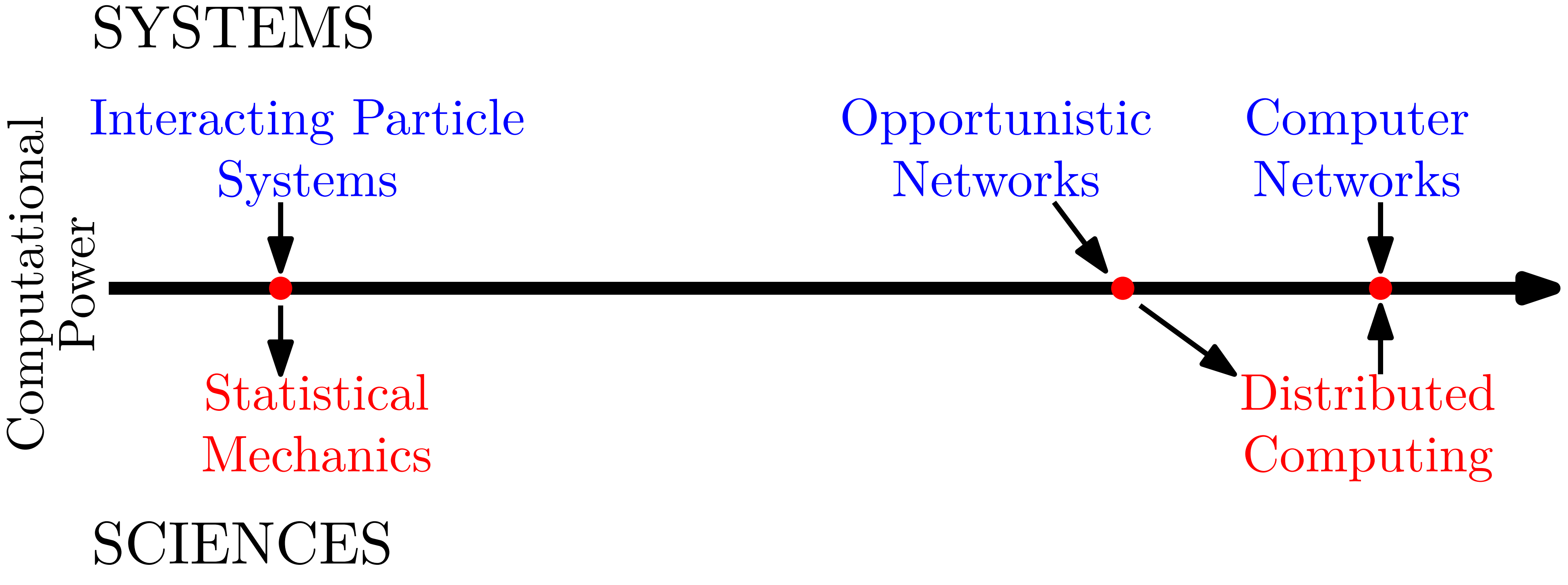

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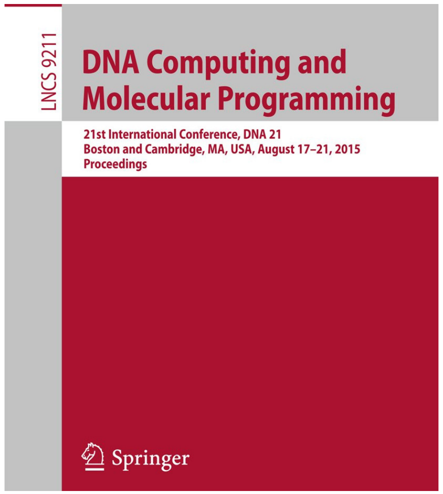
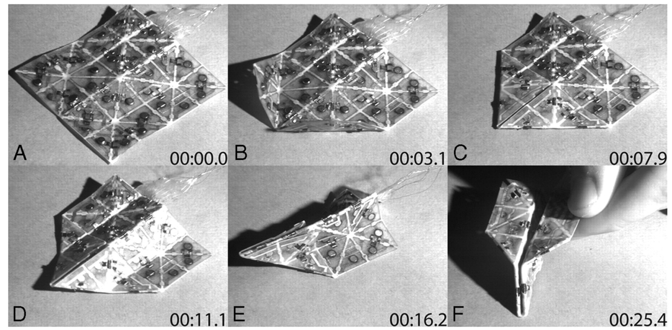
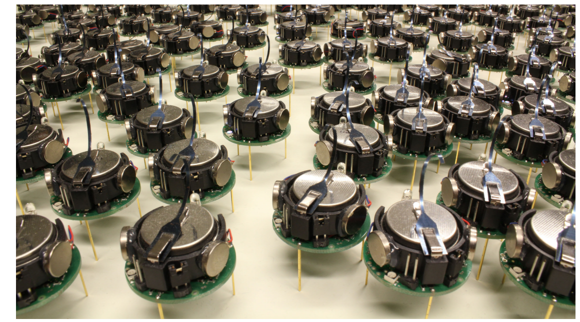
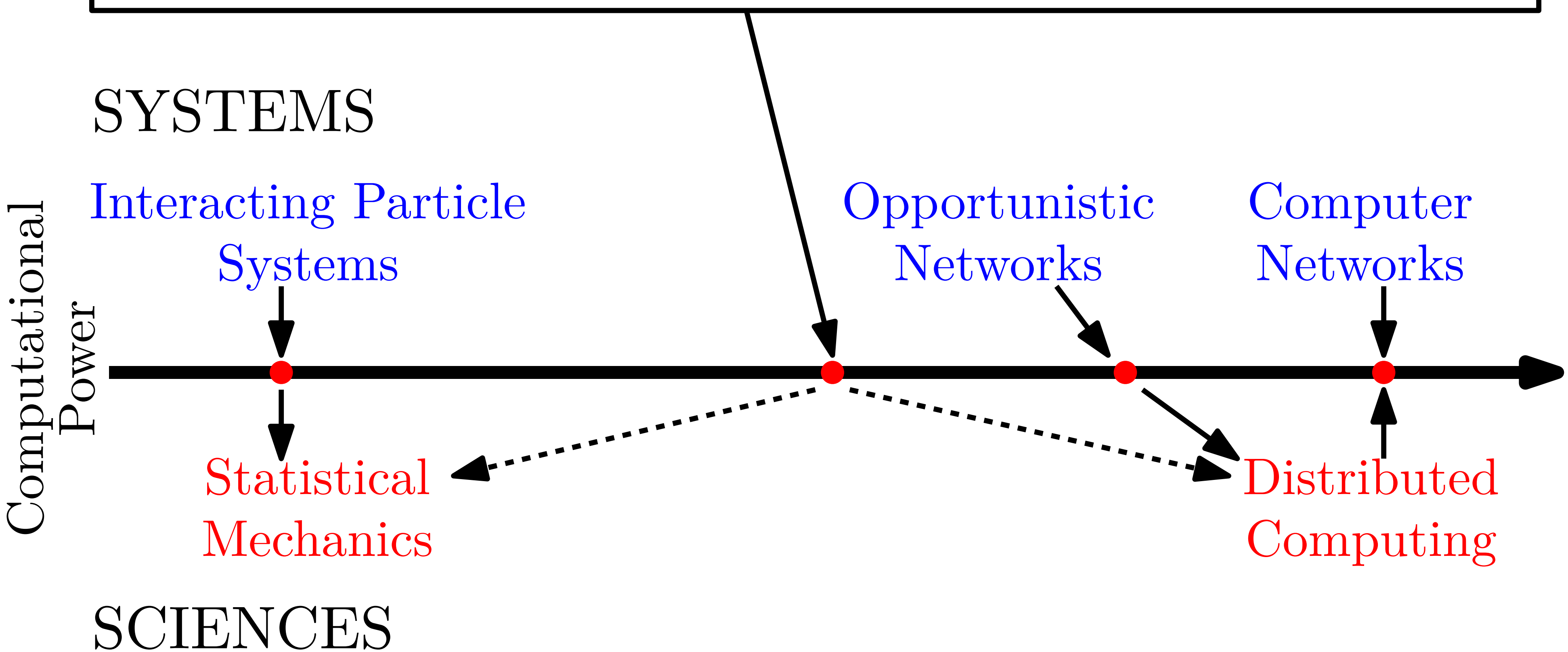






DNA/Molecular Computing, Programmable Matter, Swarms of Simple Robots



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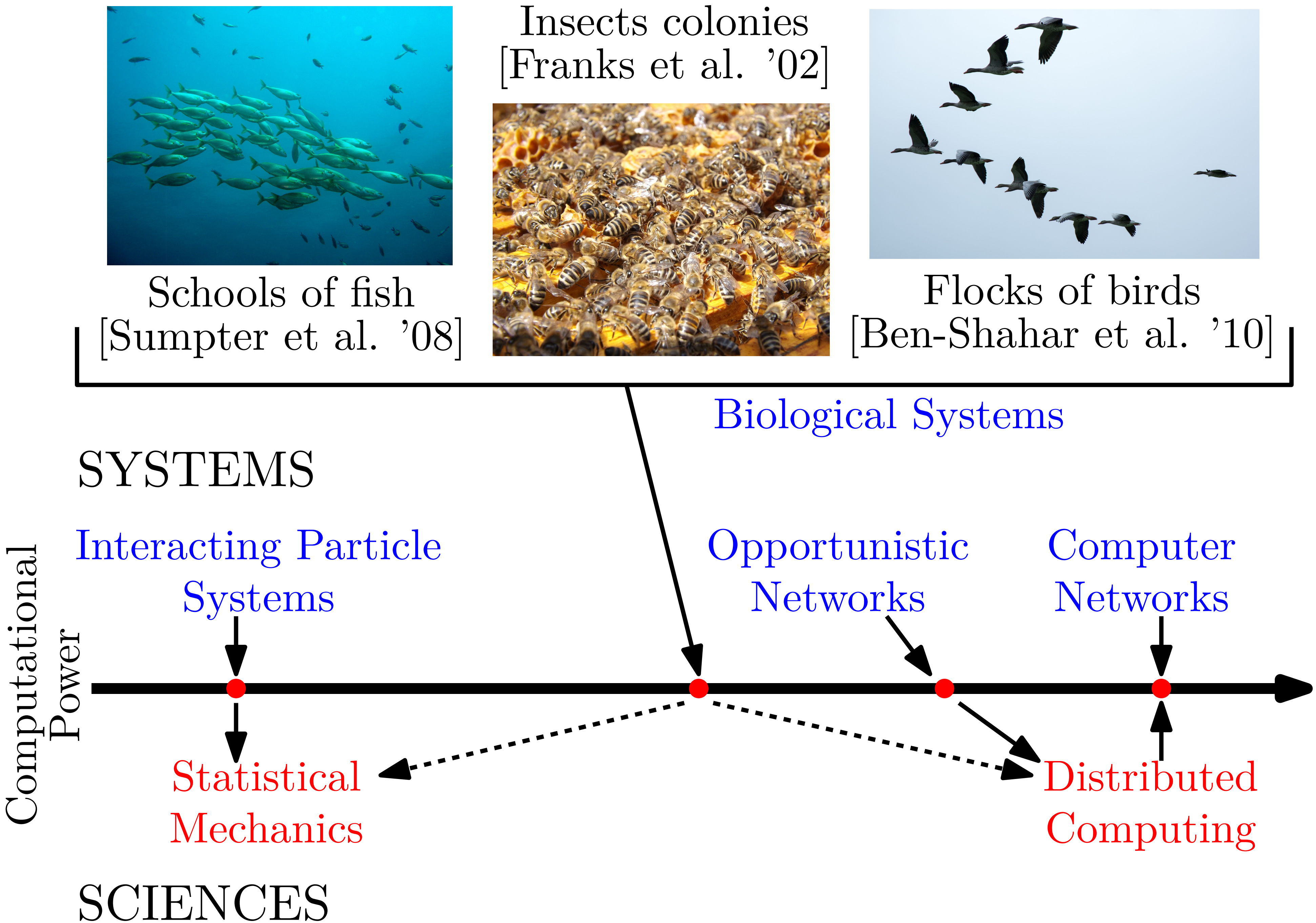

Dynamics

(informal) *Very simple* distributed algorithms: For every graph, agent and round, states are updated according to fixed rule of current state and symmetric function of states of neighbors.

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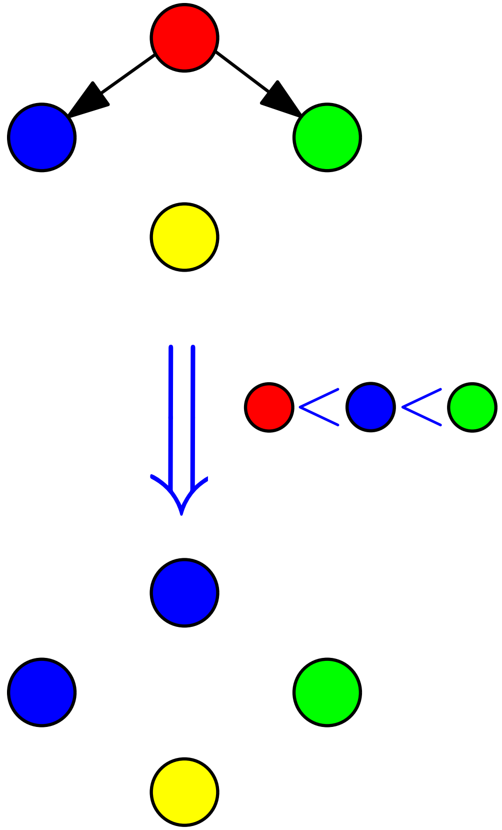
Examples of Dynamics

- 3-Median dynamics



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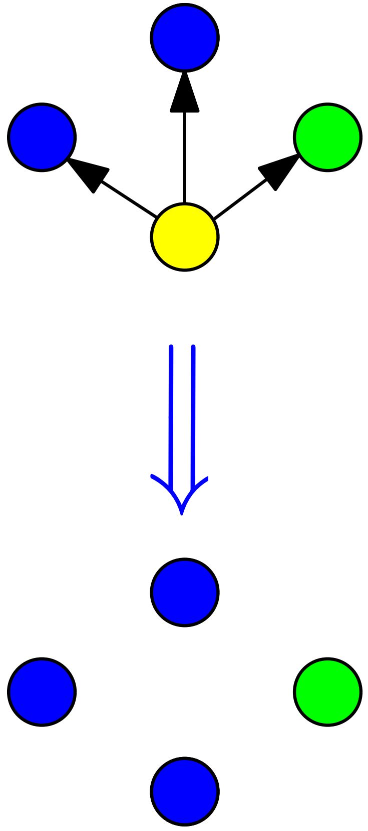
- 3-Median dynamics
- 3-Majority dynamics



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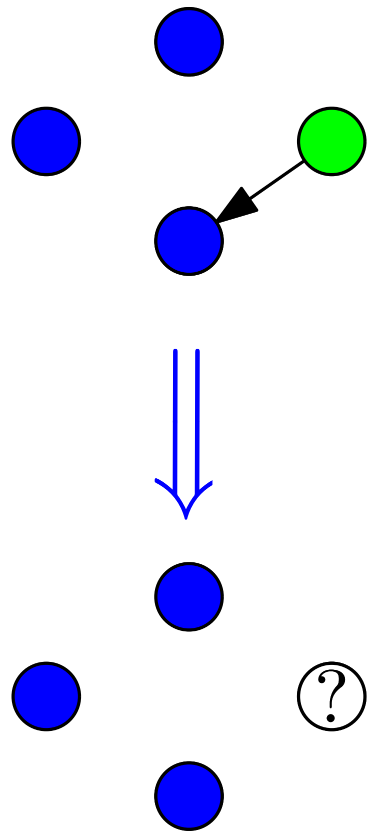

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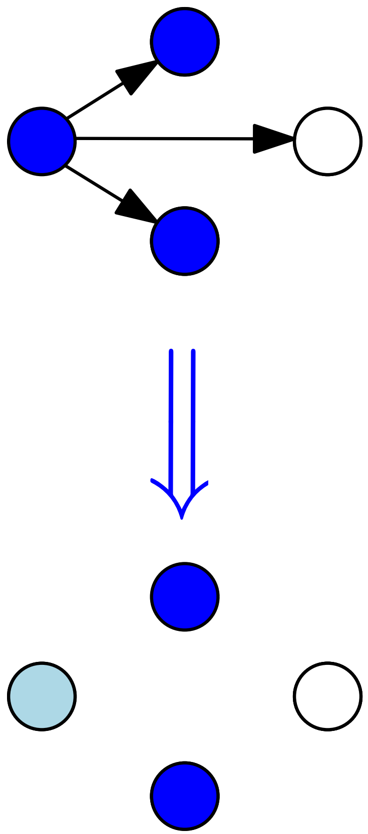
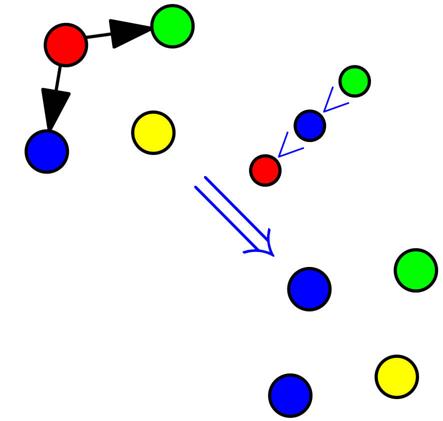


Examples of Dynamics

- 3-Median dynamics
- 3-Majority dynamics
- Undecided-state dynamics
- Averaging dynamics

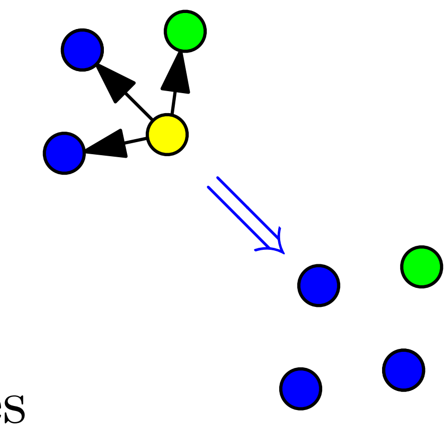
The Power of Dynamics: Plurality Consensus

Computing the Median

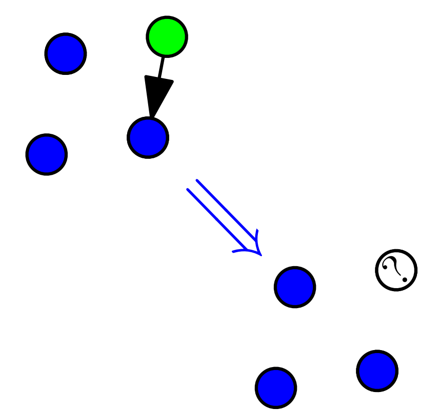


- 3-Median dynamics [Doerr et al. '11]. Converge to $\mathcal{O}(\sqrt{n \log n})$ approximation of median of system in $\mathcal{O}(\log n)$ rounds w.h.p., even if $\mathcal{O}(\sqrt{n})$ states are arbitrarily changed at each round ($\mathcal{O}(\sqrt{n})$-bounded adversary).

Computing the Majority



- 3-Majority dynamics [SPAA '14, SODA '16]. If plurality has **bias** $\mathcal{O}(\sqrt{kn \log n})$, converges to it in $\mathcal{O}(k \log n)$ rounds w.h.p., even against $o(\sqrt{n/k})$-bounded adversary. Without bias, converges in poly(k). h-majority converges in $\Omega(k/h^2)$.



- Undecided-State dynamics [SODA '15]. If majority/second-majority ($c_{maj}/c_{2^{nd}maj}$) is at least $1+\epsilon$, system converges to plurality within $\tilde{\Theta}(\sum_{i=1}^{k} \left(c_i^{(0)}/c_{maj}^{(0)}\right)^2)$ rounds w.h.p.

The Median, the Mode and... the Mean

Dynamics can solve Consensus, Median, Majority, in robust and fault tolerant ways, but this is trivial in centralized setting.

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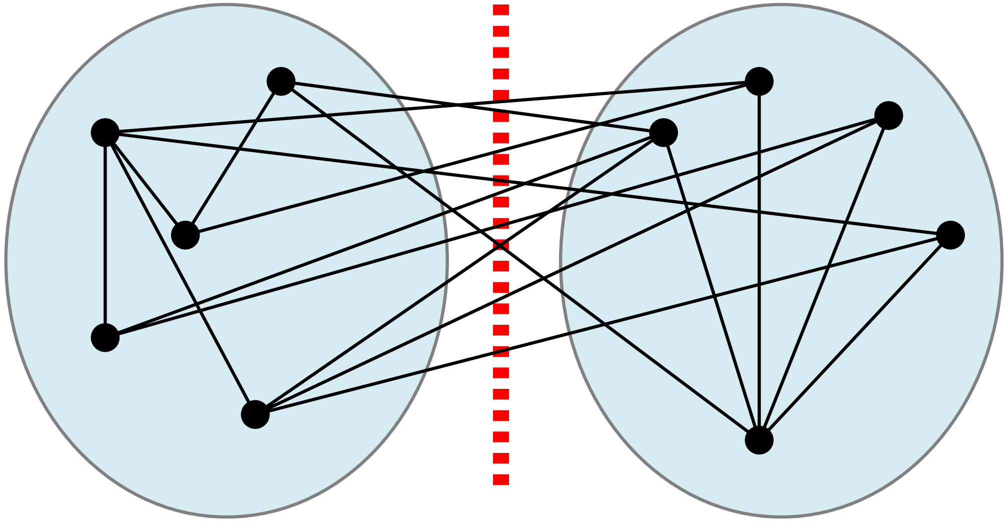
Can dynamics solve a problem non-trivial in centralized setting?

Community Detection as Minimum Bisection

Minimum Bisection Problem.

Input: a graph G with $2n$ nodes.

Output: $S = \arg\min_{\substack{S \subset V \\ |S|=n}} E(S, V - S)$.

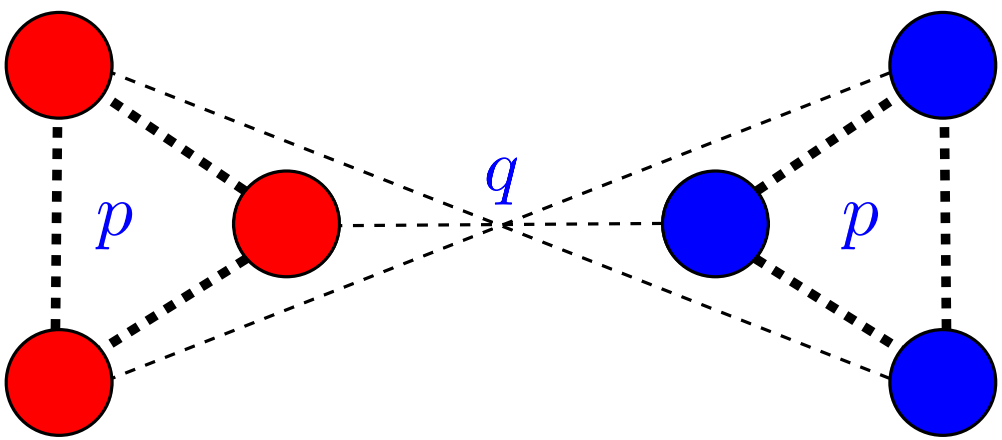


[Garey, Johnson, Stockmeyer '76]:

Min-Bisection is *NP-Complete*.

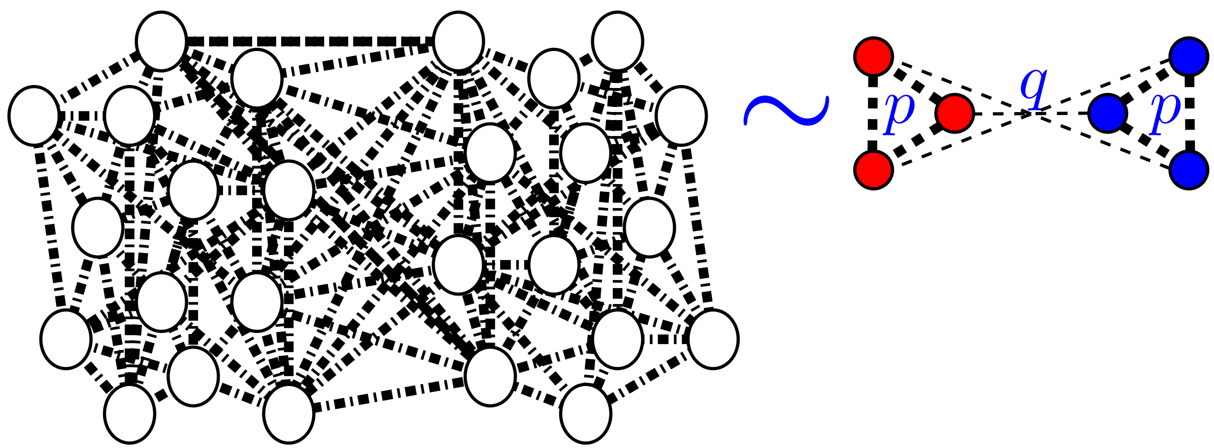
The Stochastic Block Model

Stochastic Block Model (SBM). Two "communities" of equal size V_1 and V_2, each edge inside a community included with probability $p = \frac{a}{n}$, each edge across communities included with probability $q = \frac{b}{n} < p$.



The Stochastic Block Model

Reconstruction problem. Given graph generated by SBM, find original partition.



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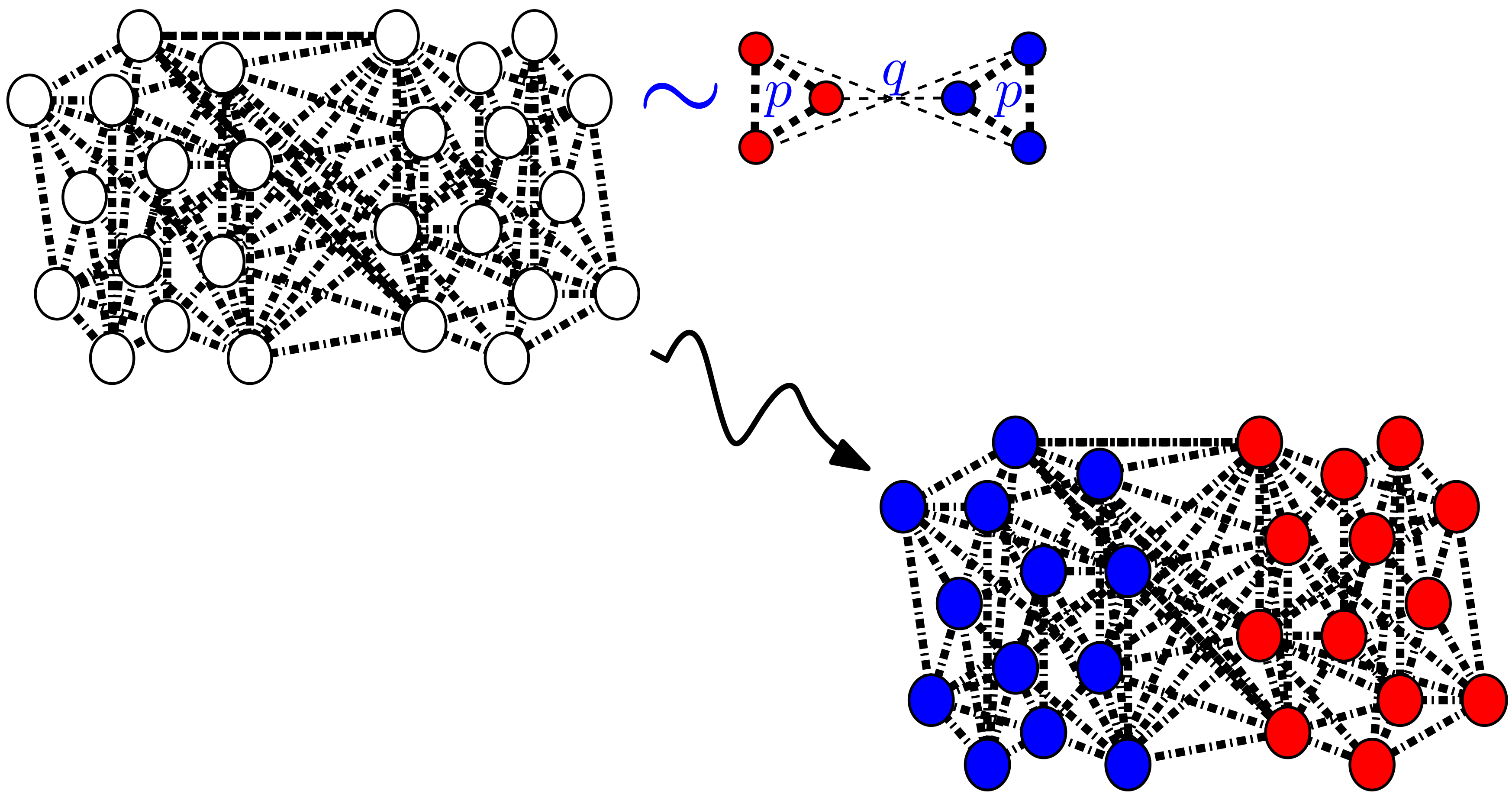
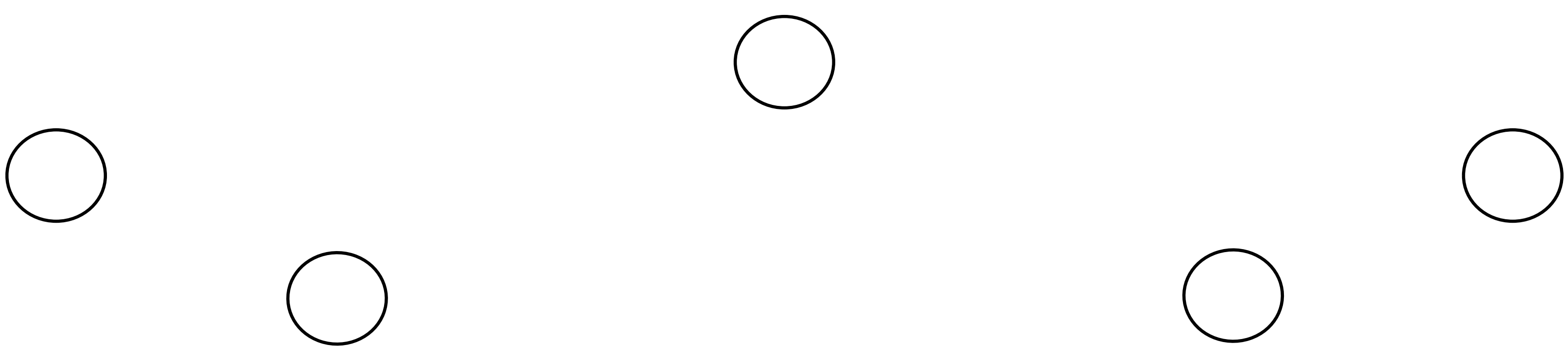


The Averaging Dynamics in the $\mathcal{LOCAL}$ Model

Al nodes at the same time:

- At $t = 0$, randomly pick value $x^{(t)} \in \{\mathbf{+1}, \mathbf{-1}\}$.
- Then, at each round
  1. Set value $x^{(t)}$ to average of neighbors,
  2. Set label to **blue** if $x^{(t)} < x^{(t-1)}$, **red** otherwise.



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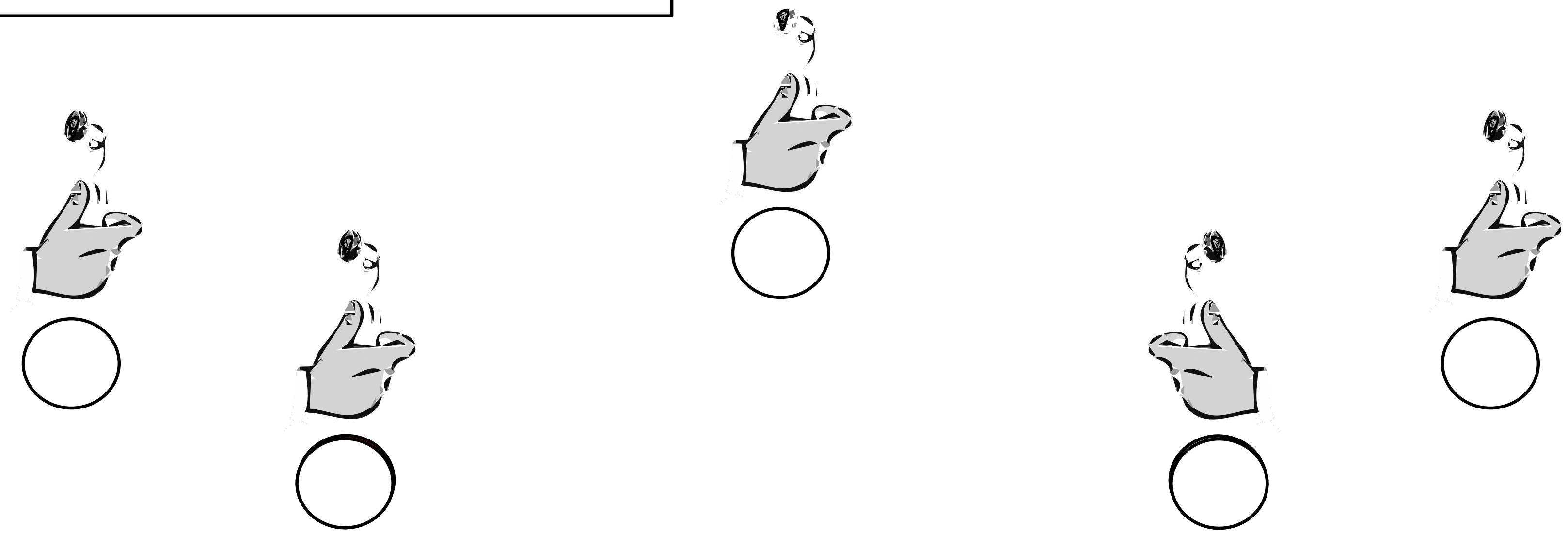

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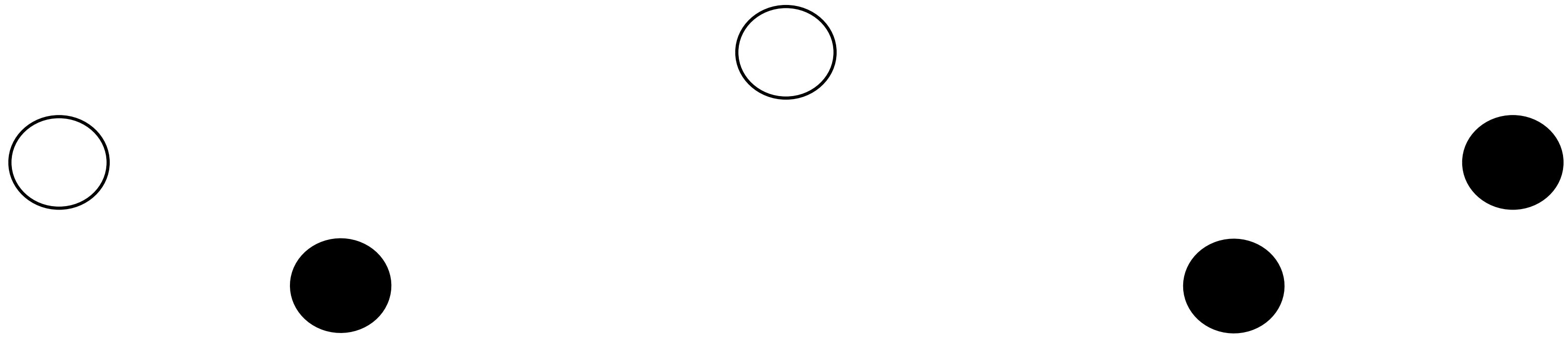

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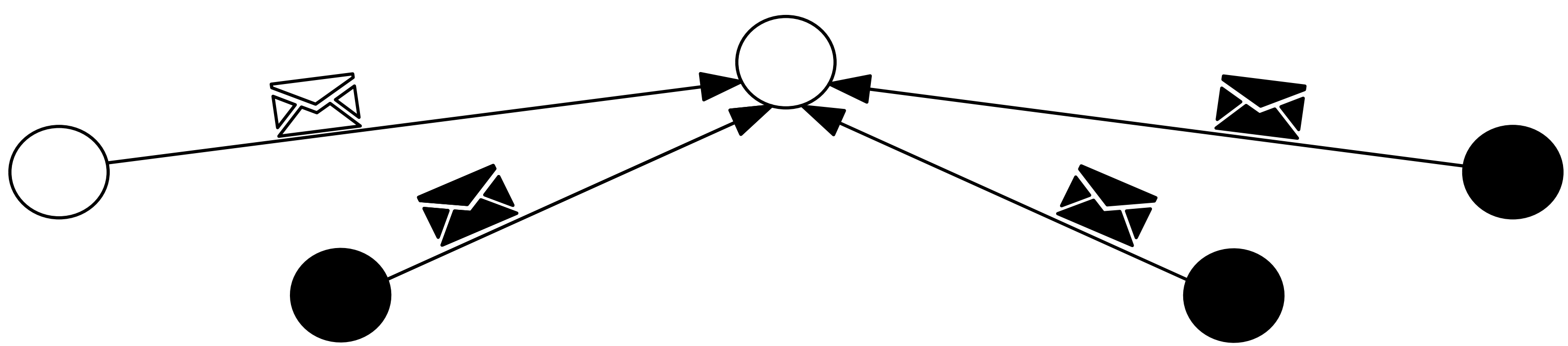

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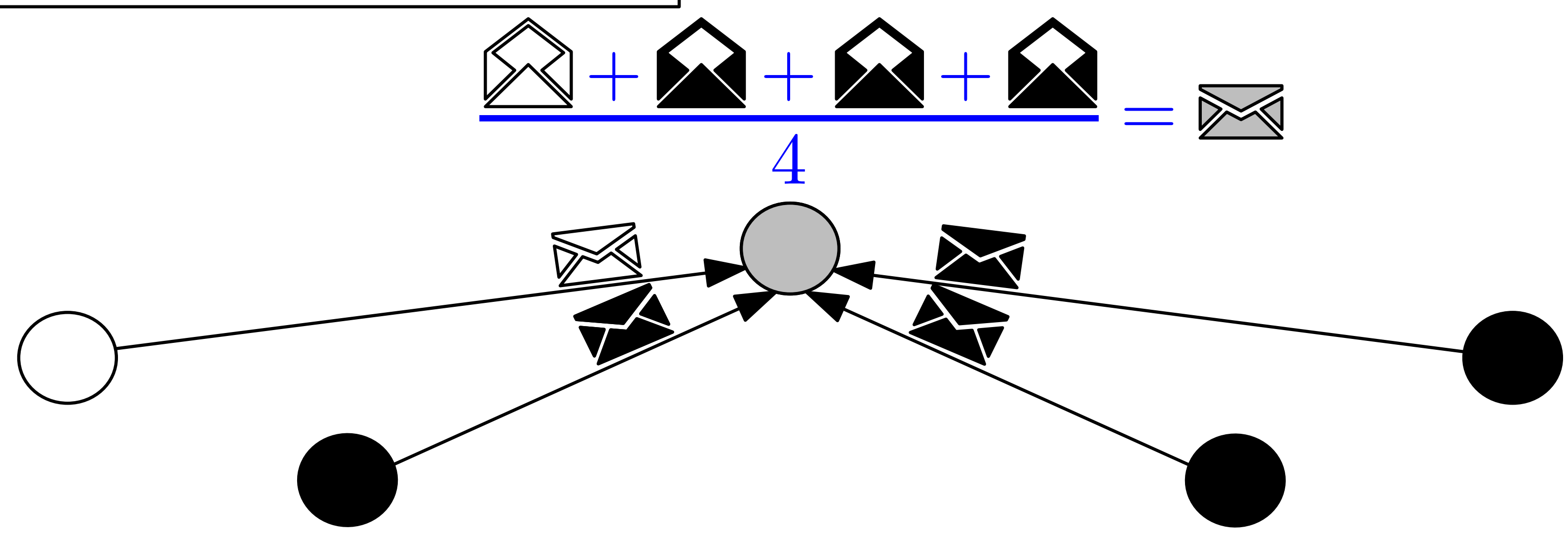

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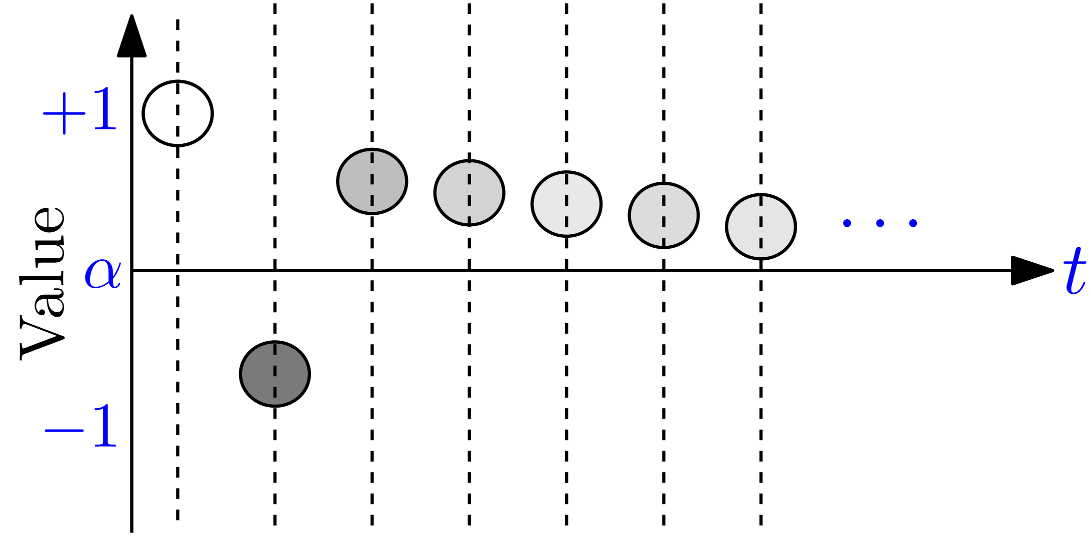

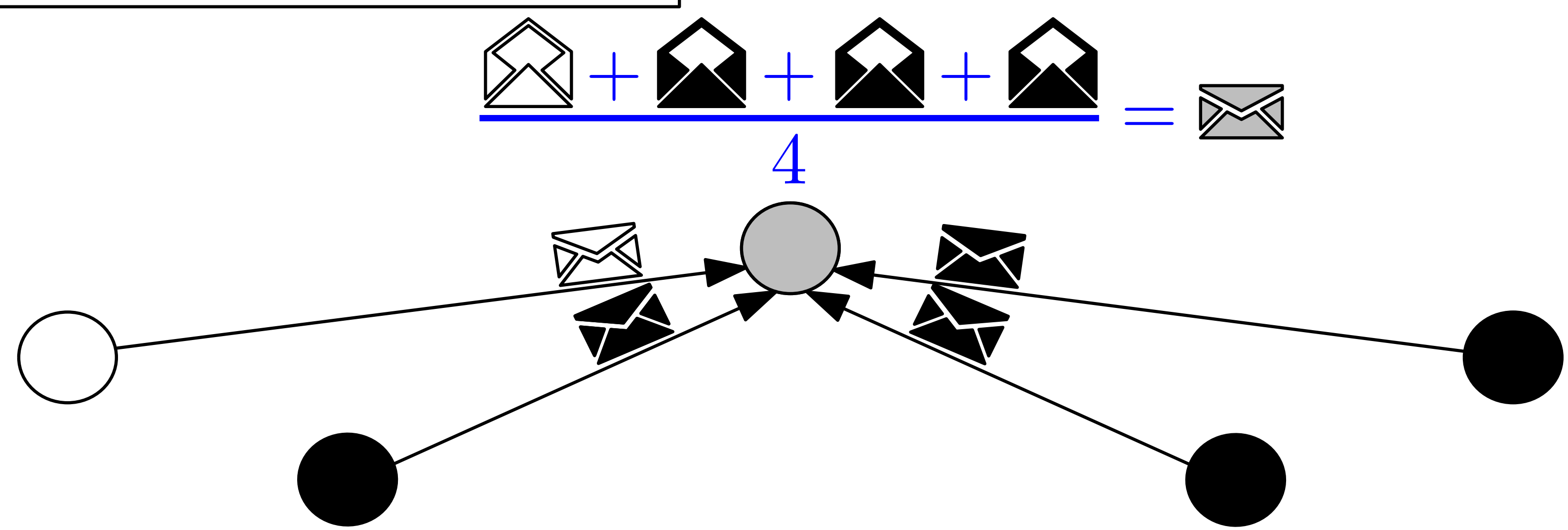

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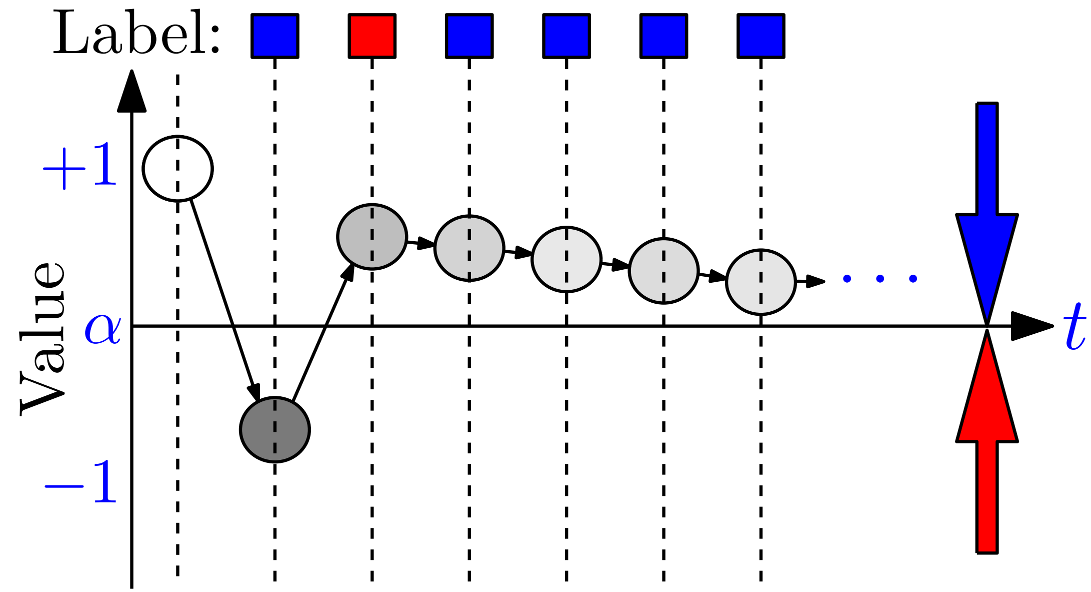

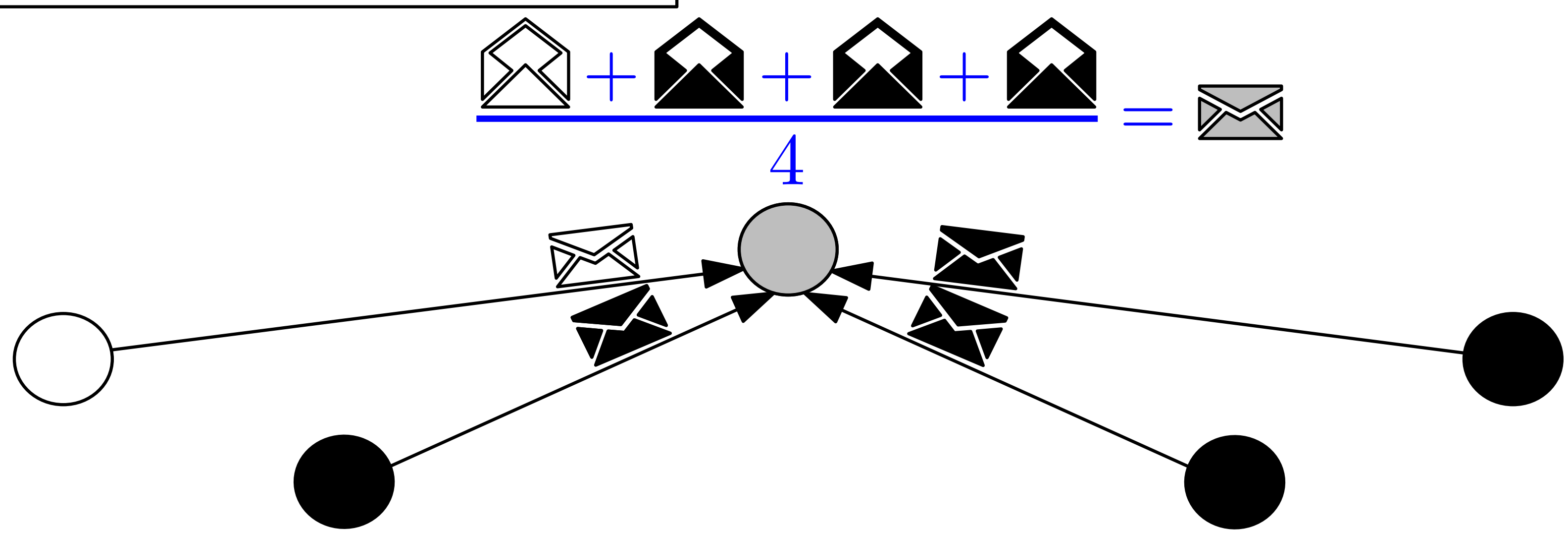

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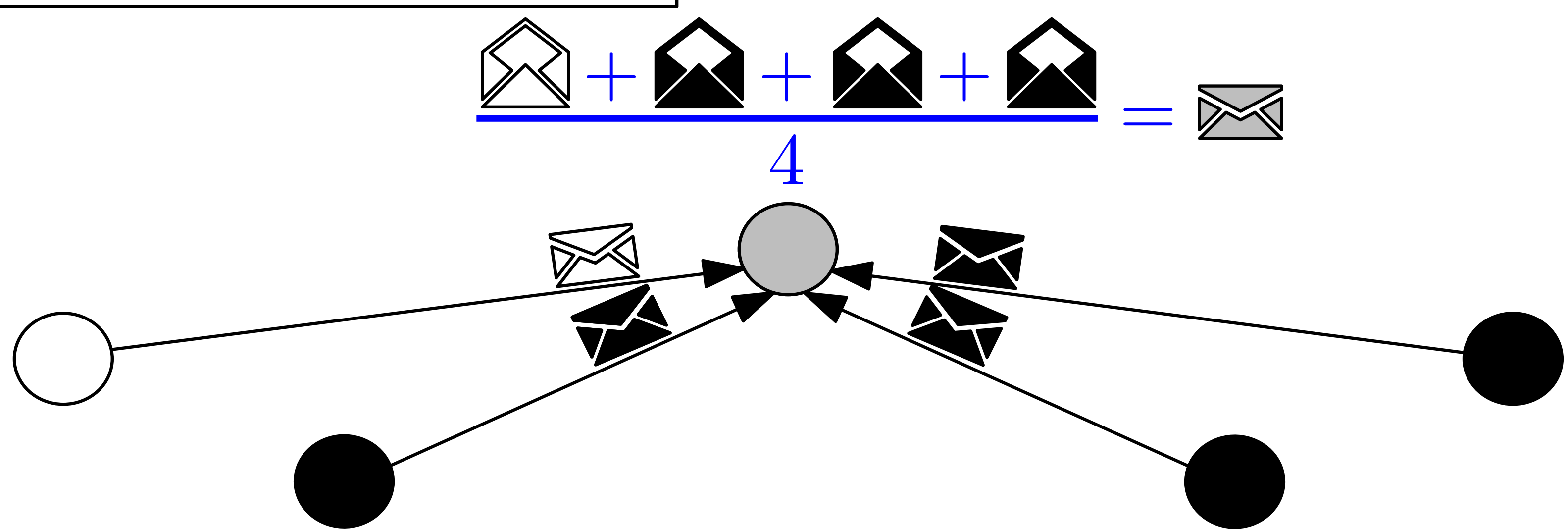
Well studied process [Shah '09]:

- Converges to (weighted) global average of initial values,
- Convergence time $=$ mixing time of G,
- Important applications in fault-tolerant self-stabilizing consensus.



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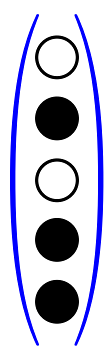
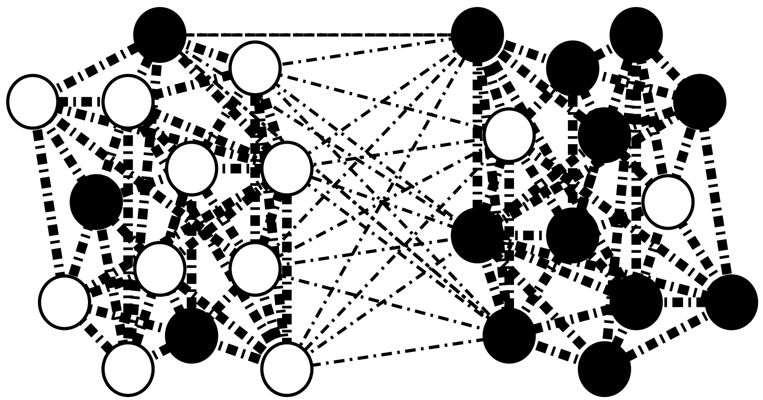
Averaging is a **linear** dynamics

$\mathbf{x}^{(t)} =$ 

$$\mathbf{x}^{(t)} = P \cdot \mathbf{x}^{(t-1)} = P^t \cdot \mathbf{x}^{(0)}$$

P transition matrix of random walk

Community Detection via Averaging Dynamics



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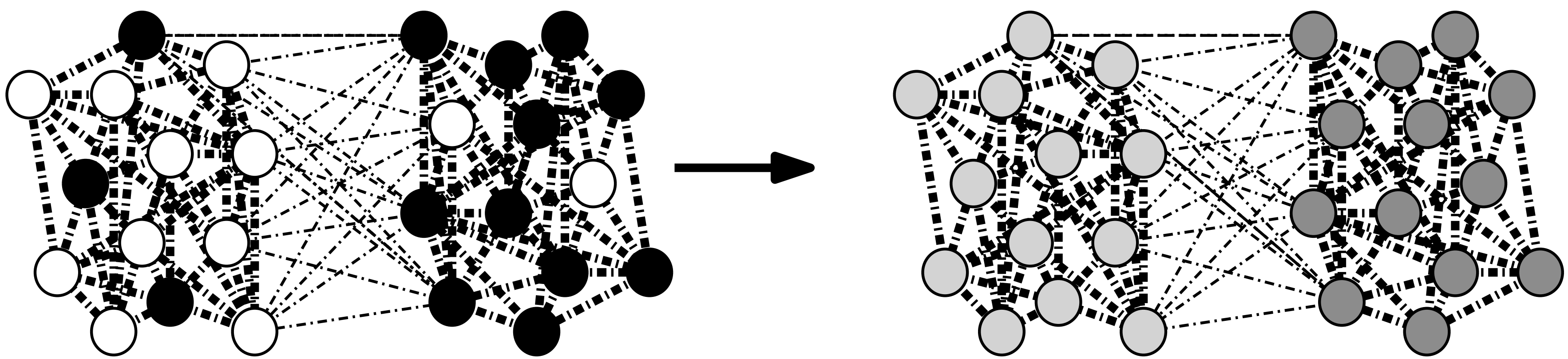

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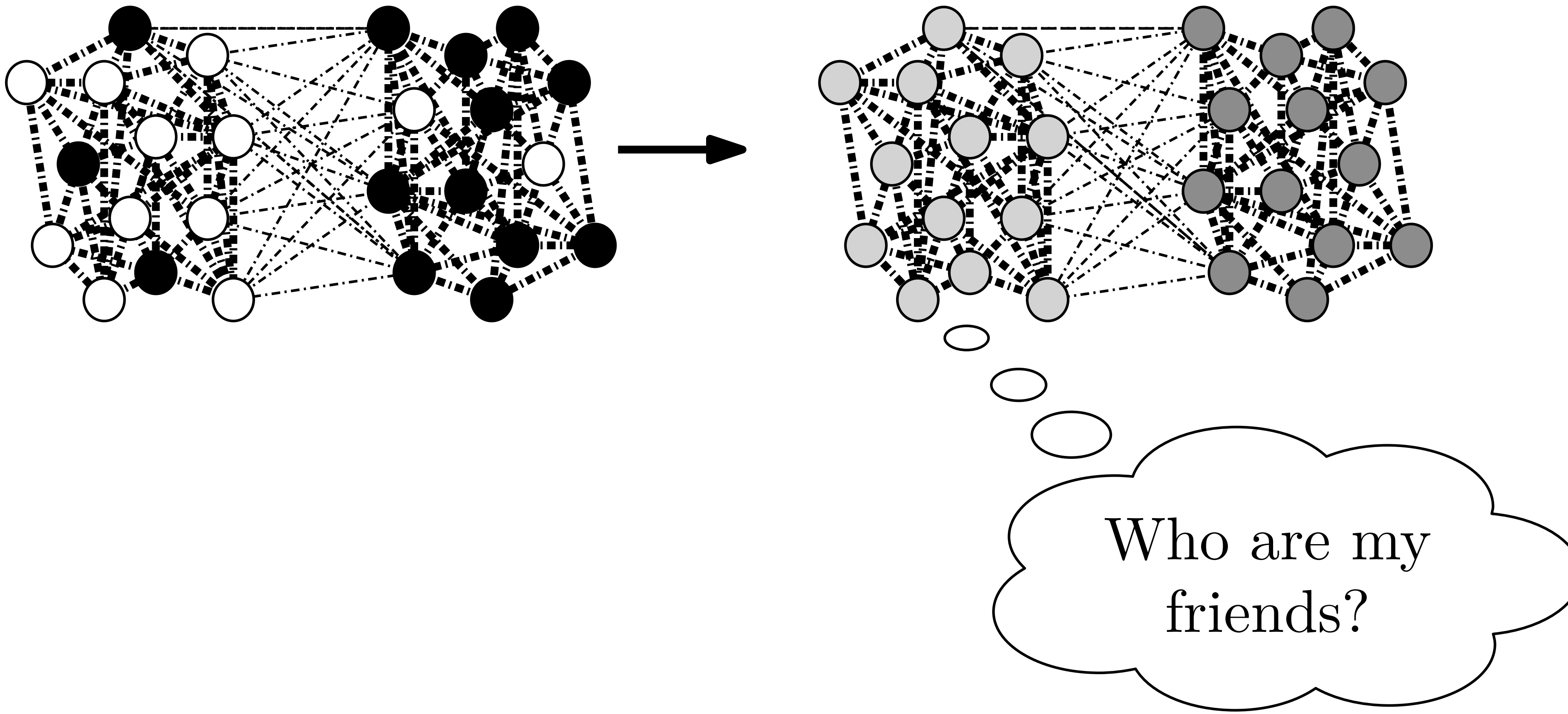

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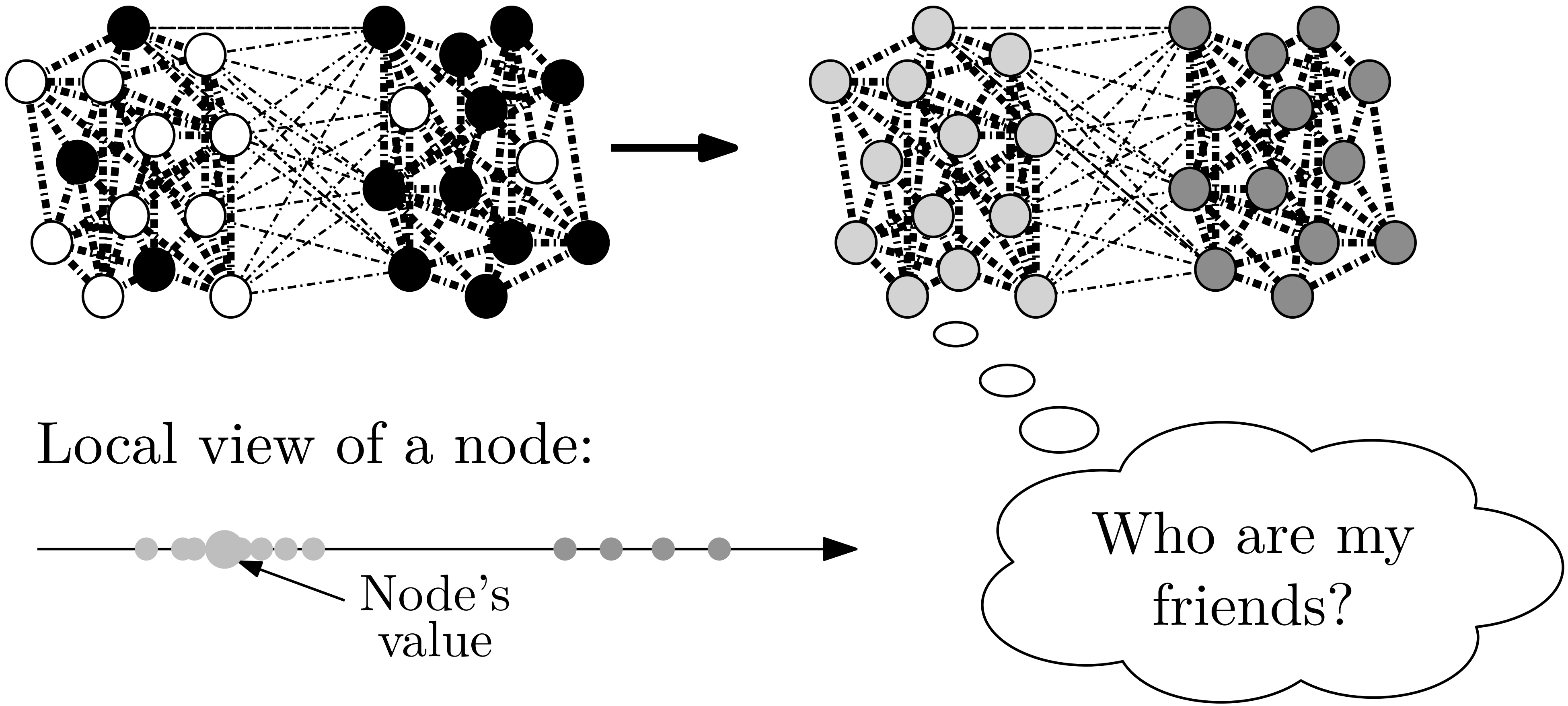

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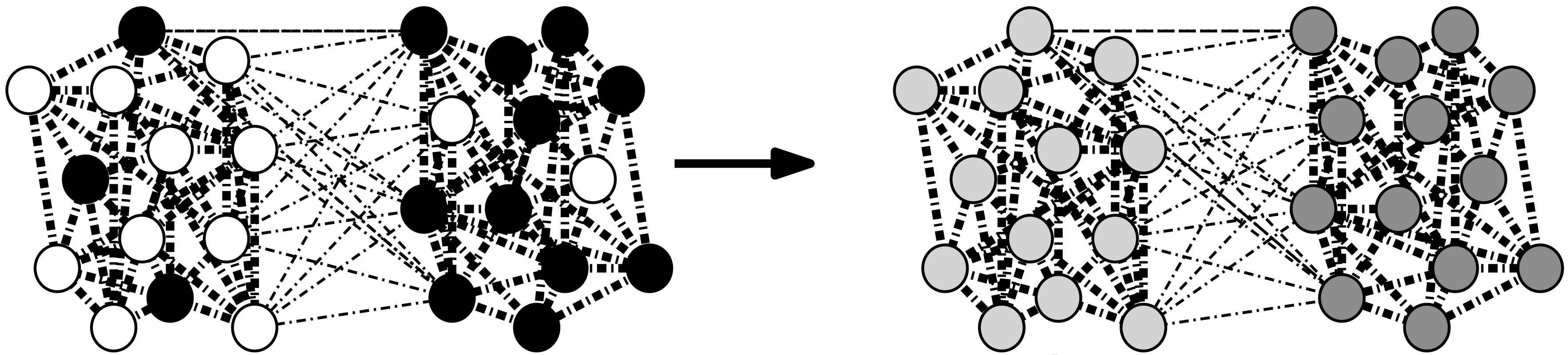
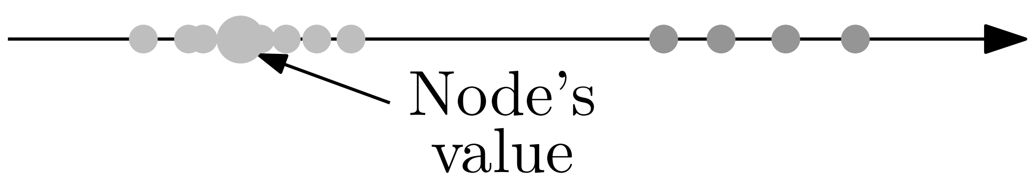


Local view of a node:



Irregular case:

- *outliers*?
- *no neighbors* in the other community?

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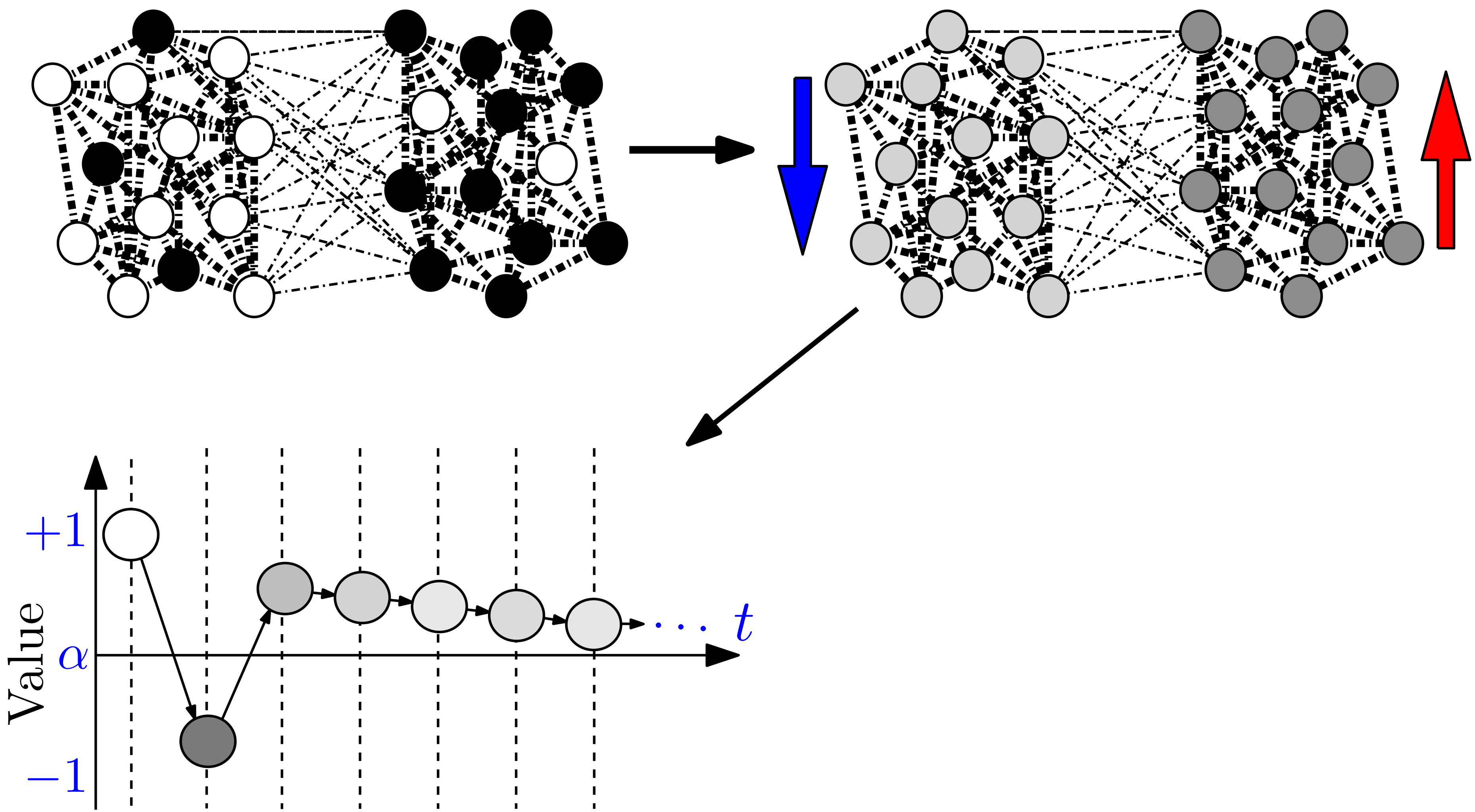

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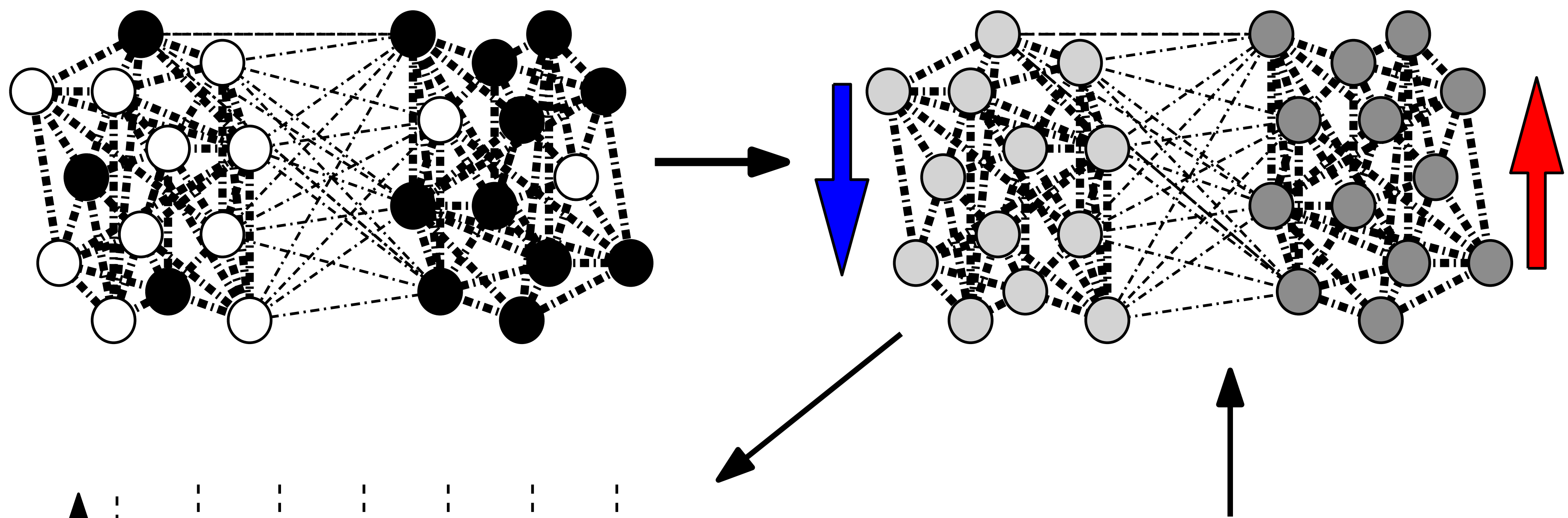
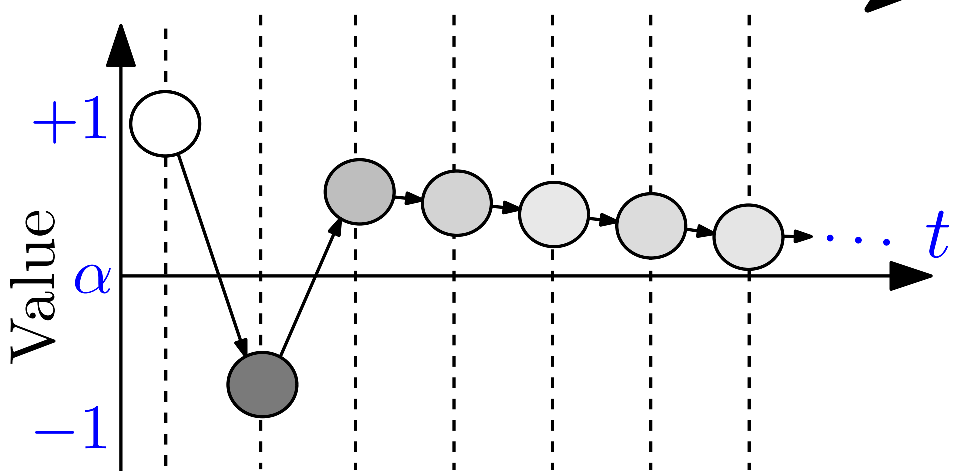




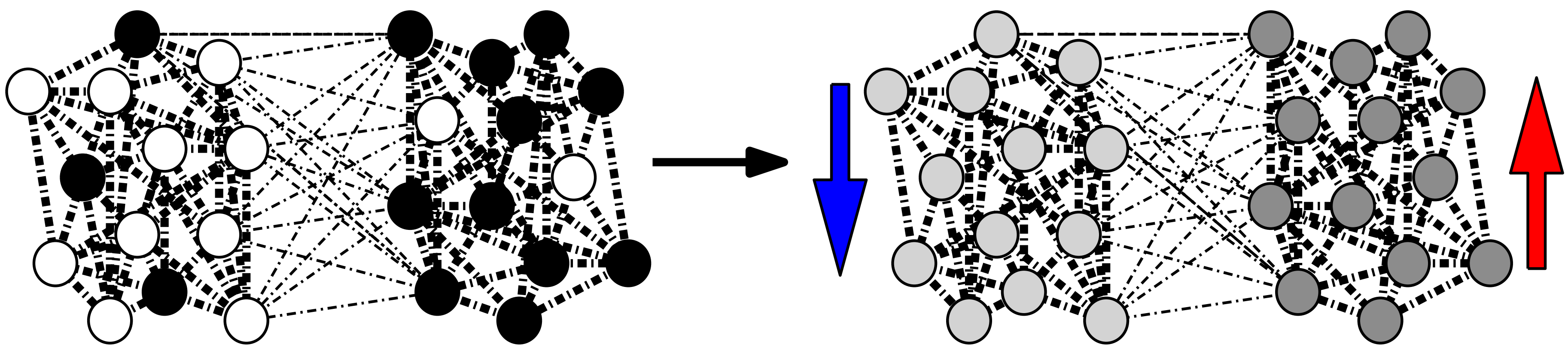
$v_1, \ldots, v_n$ eigenvectors of random walk matrix P:

$$v_1 = \mathbb{1} = (1, \ldots, 1)$$

$$v_2 \approx \chi = (1, \ldots, 1, -1, \ldots, -1)$$

"nice" graph

Community Detection via Averaging Dynamics

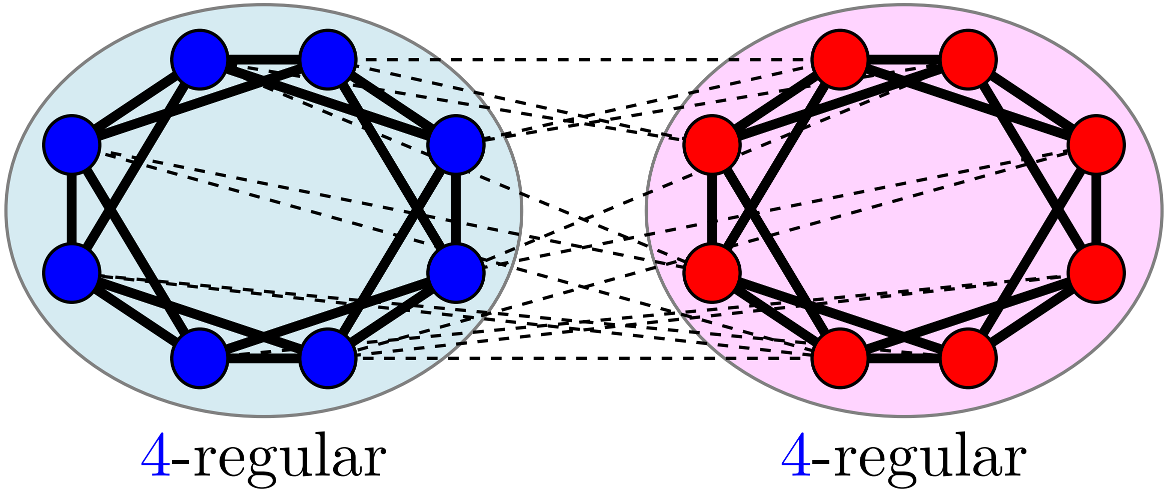


[SODA '17](Informal). $G = (V_1 \dot{\bigcup} V_2, E)$ s.t.
i) $\chi = \mathbf{1}_{V_1} - \mathbf{1}_{V_2}$ close to right-eigenvector of eigenvalue λ_2 of transition matrix of G, and
ii) gap between λ_2 and $\lambda = \max\{\lambda_3, |\lambda_n|\}$ sufficiently large,
then Averaging (approximately) identifies (V_1, V_2).

Toy Case: Regular Stochastic Block Model

Regular SBM (RSBM) [Brito et al. SODA'16]. A graph $G = (V_1 \dot{\bigcup} V_2, E)$ s.t.

- $|V_1| = |V_2|$,
- $G\big|_{V_1}, G\big|_{V_2} \sim$ random a-regular graphs
- $G\big|_{E(V_1,V_2)} \sim$ random b-regular bipartite graph.



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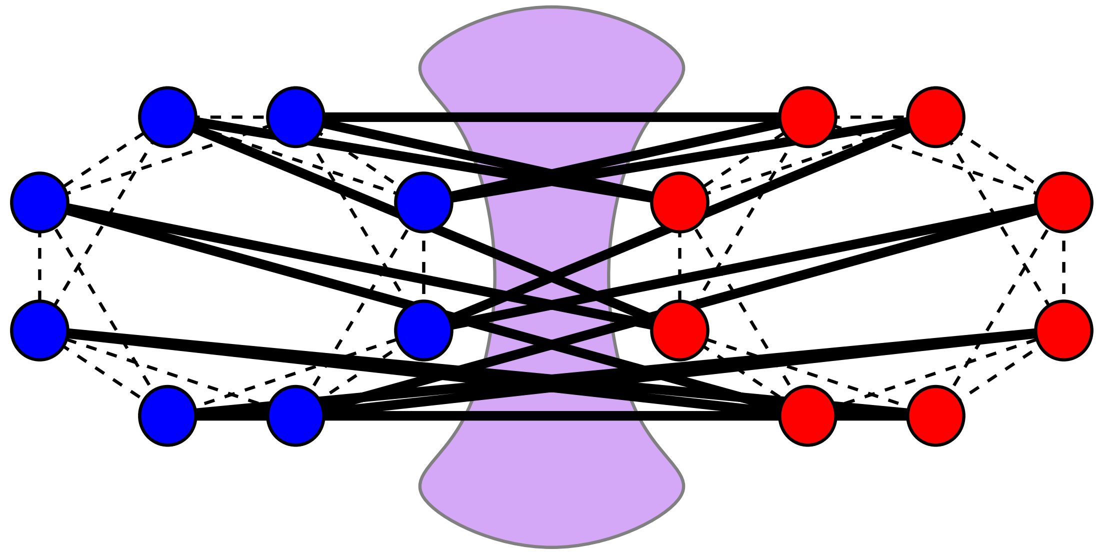

2-regular bipartite

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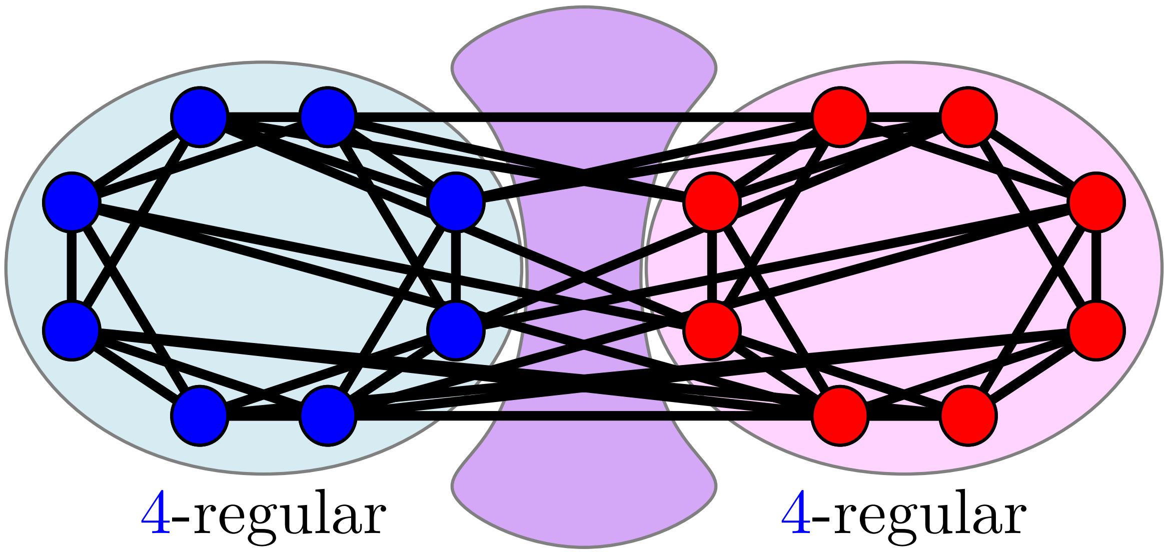

Analysis on Regular SBM

$P \longrightarrow$ symmetric $\Longrightarrow$ orthonormal eigenvectors $\mathbf{v}_1, ..., \mathbf{v}_n$ and real eigenvalues $\lambda_1, ..., \lambda_n$.

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Regular SBM $\Longrightarrow$ $P \frac{1}{\sqrt{n}} \chi = \left(\frac{a-b}{a+b}\right) \cdot \frac{1}{\sqrt{n}} \chi$

$$\frac{1}{a+b} \left(\begin{array}{c|c} \cdots\cdots\cdots & \cdots\cdots\cdots \\ \cdots a \text{ “1”s} \cdots & \cdots b \text{ “1”s} \cdots \\ \cdots\cdots\cdots & \cdots\cdots\cdots \\ \hline \cdots\cdots\cdots & \cdots\cdots\cdots \\ \cdots b \text{ “1”s} \cdots & \cdots a \text{ “1”s} \cdots \\ \cdots\cdots\cdots & \cdots\cdots\cdots \end{array} \right) \cdot \begin{pmatrix} 1 \\ \vdots \\ 1 \\ -1 \\ \vdots \\ -1 \end{pmatrix} = \frac{a-b}{a+b} \begin{pmatrix} 1 \\ \vdots \\ 1 \\ -1 \\ \vdots \\ -1 \end{pmatrix}$$

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Regular SBM $\implies$ $P \frac{1}{\sqrt{n}} \chi = \left(\frac{a-b}{a+b}\right) \cdot \frac{1}{\sqrt{n}} \chi$

W.h.p. $\max\{\lambda_3, |\lambda_n|\}(1+\delta) < \frac{a-b}{a+b} = \lambda_2$, then

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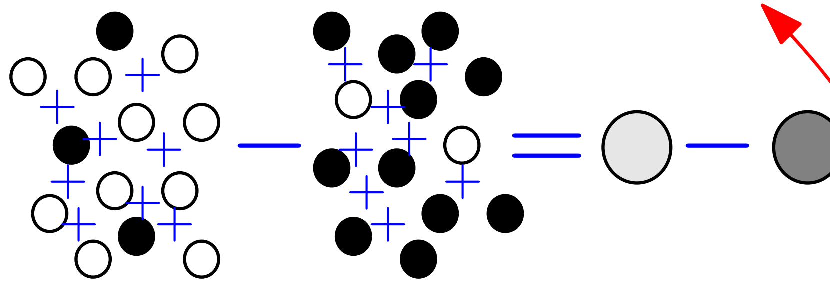
with $\|\mathbf{e}^{(t)}\| \leq (\max\{\lambda_3, |\lambda_n|\})^t \sqrt{n}$

Analysis on Regular SBM

$\frac{1}{n}\sum_{u\in V_1}\mathbf{x}^{(0)}(u) - \frac{1}{n}\sum_{u\in V_2}\mathbf{x}^{(0)}(u)$



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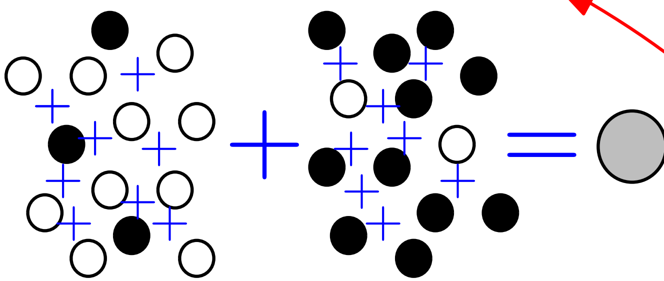

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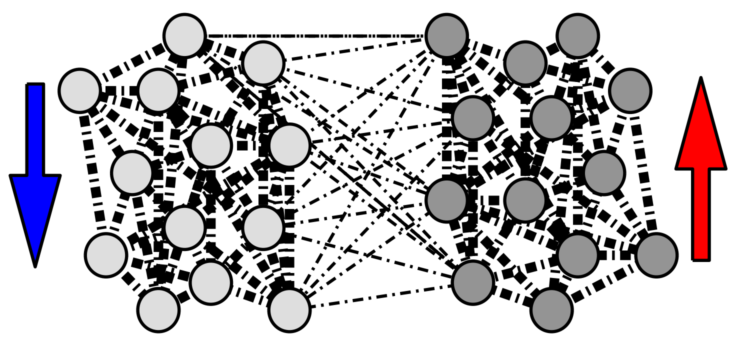

$$\boxed{\text{sign}(\mathbf{x}^{(t)}(u) - \mathbf{x}^{(t-1)}(u)) \propto \text{sign}(\chi(u))}$$

Sparsification of the Averaging Dynamics

Averaging Dynamics in $\mathcal{LOCAL}$ Model:
$\mathcal{O}(d)$ messages per round :-(

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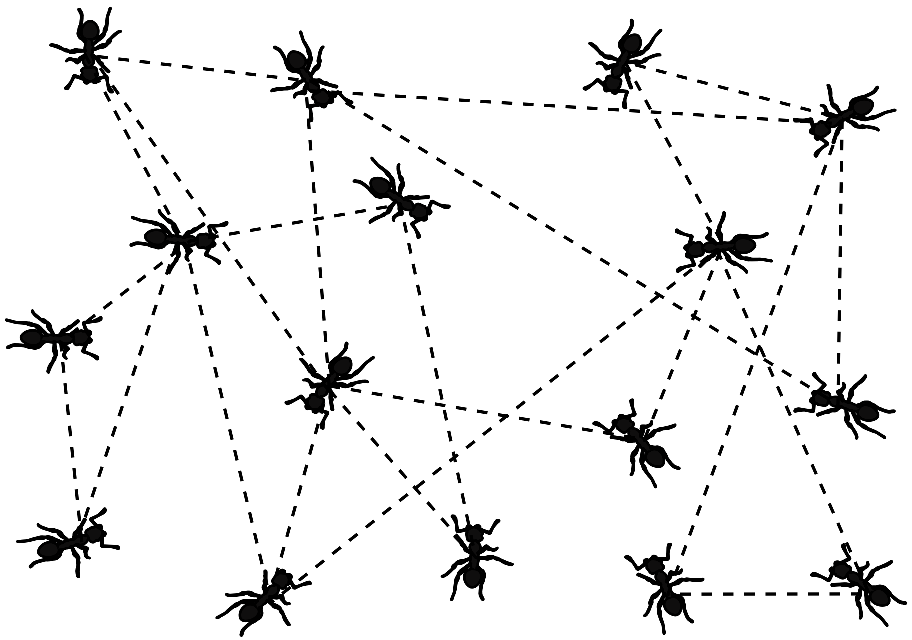
Problem: no concentration tools for matrix *products*
(e.g. no logarithm for noncommutative matrices)

Communication Model: Population Protocol

Population protocol: at each round a random edge is chosen and the two corresponding agent interact.



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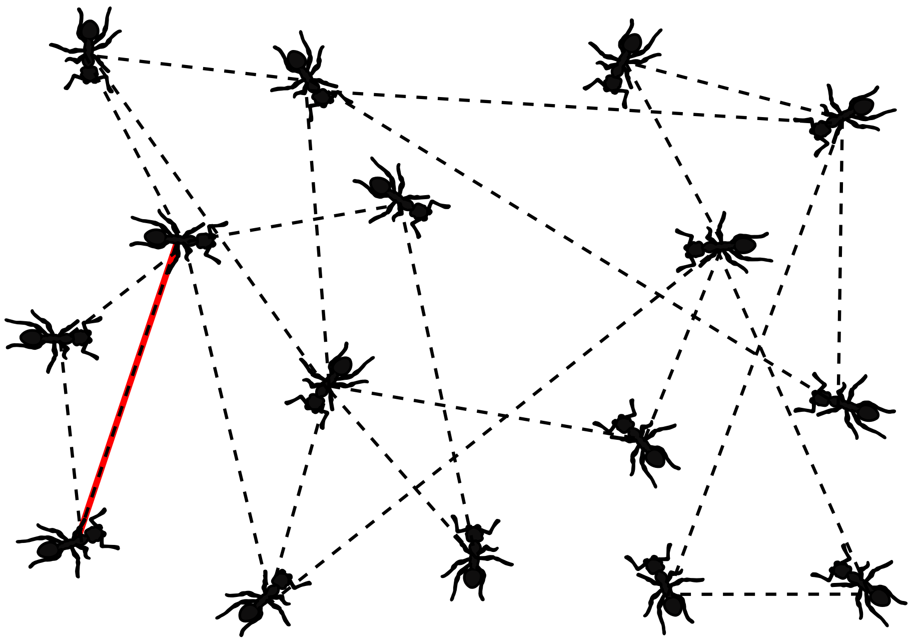

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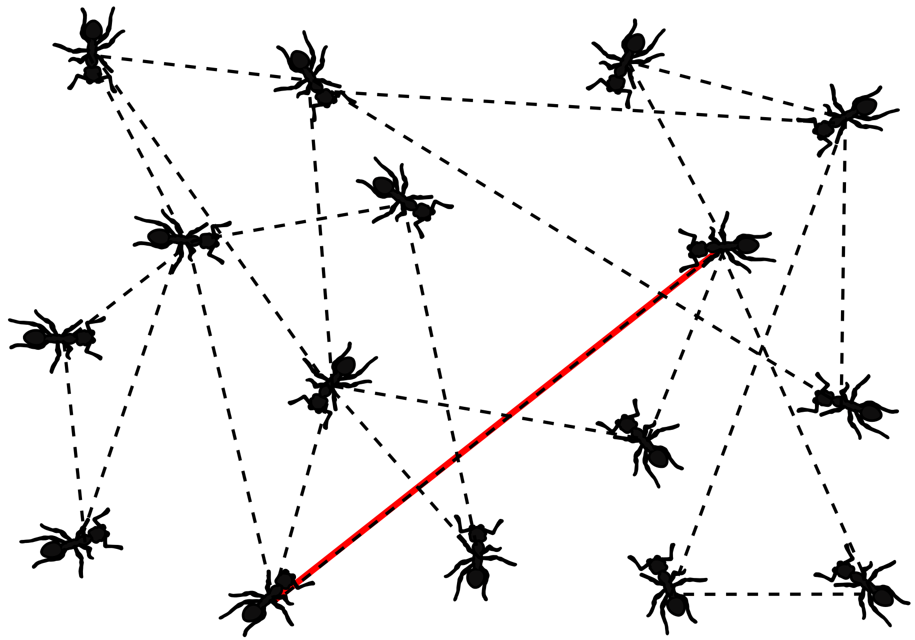

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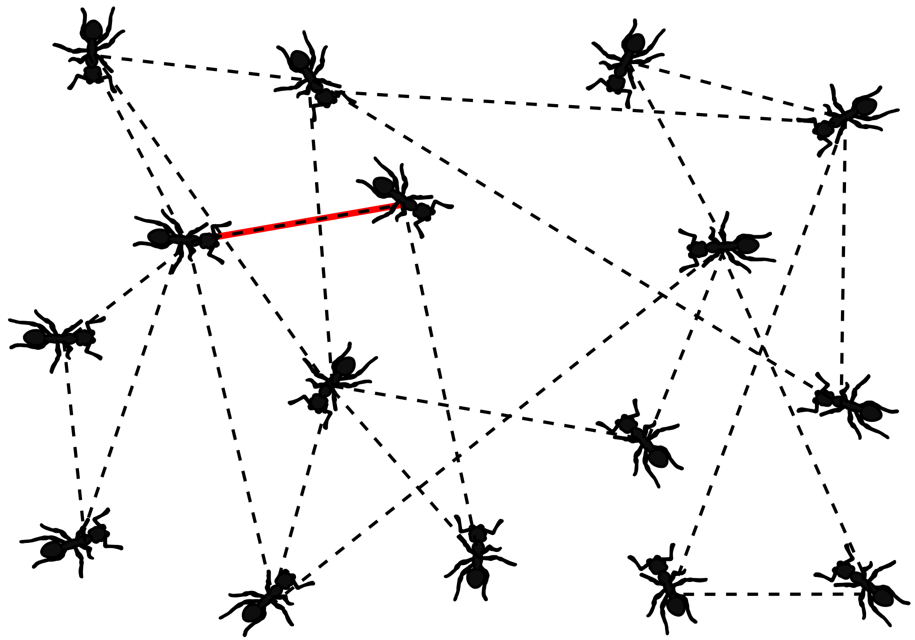

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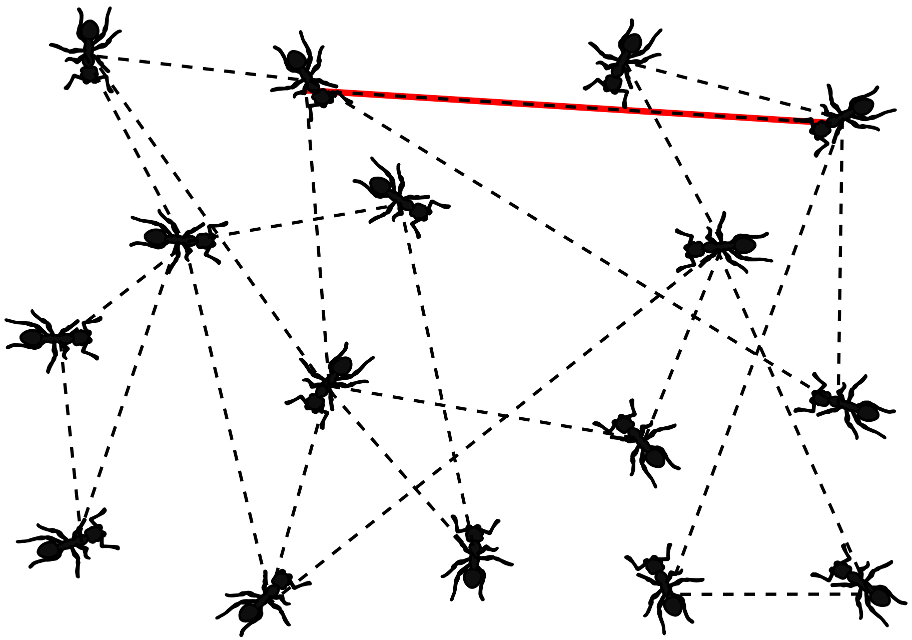

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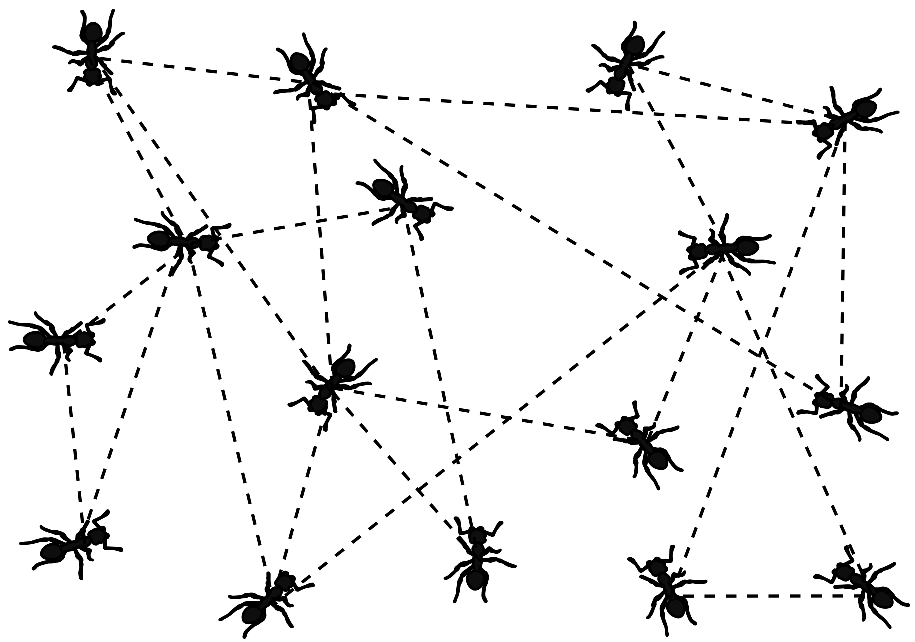
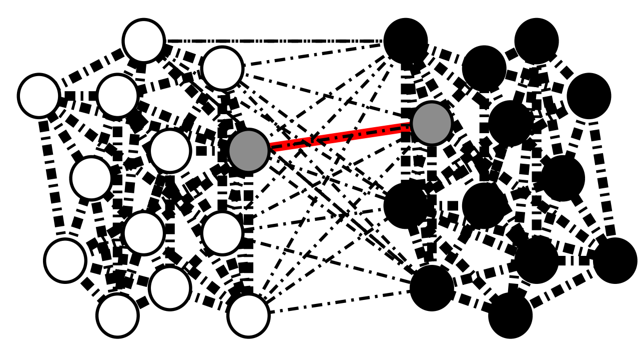


!!!: The *variance* of picking a random edge breaks the monotonicity and seems to prevent concentration.



Community Sensitive Labeling

CSL(m,T):

- At the outset
$$\mathbf{x}_u^{(0)} \sim \mathrm{Unif}(\{-1,+1\}^m).$$
- In each round, the endpoints of the random edge choose a random index $j \in [m]$ and set
$$\mathbf{x}_u(j) = \mathbf{x}_v(j) = \frac{\mathbf{x}_u(j) + \mathbf{x}_v(j)}{2};$$ (cfr [Boyd et al. '06]).
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Thm. $G = (V_1 \dot{\bigcup} V_2, E)$ regular SBM s.t. $d\epsilon^4 \gg b \log^2 n$, then CSL($m$,$T$) with $m = \Theta(\epsilon^{-1} \log n)$ and $T = \Theta(\log n)$ labels all nodes but a set U with size $|U| \leq \sqrt{\epsilon} n$, in such a way that

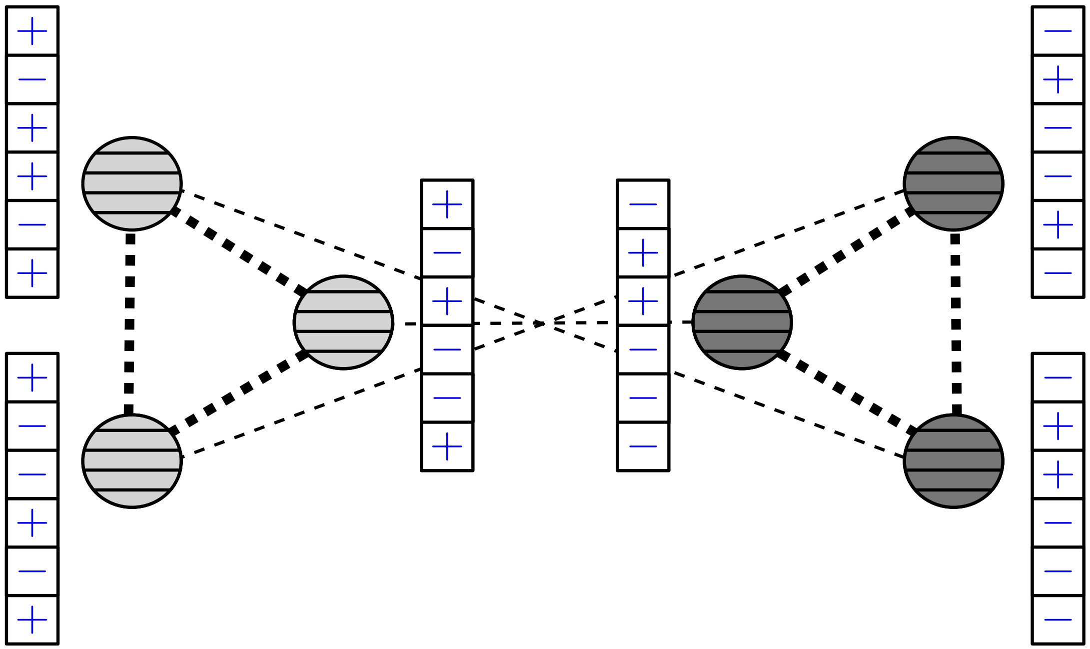
- the labels of nodes in the same community agree on at least $5/6$ entries, and
- the labels of nodes in different communities differ in more than $1/6$ entries.

Community Sensitive Labeling

Example:

> 2 different labels
$\Longrightarrow$ foes!

≤ 2 different labels
$\Longrightarrow$ friends!



Warning: not a dynamics!

Analysis 1/4

Proof Ingredient 1. We are done if, for any fixed component j, all *lucky* nodes $u \notin U$ are such that

$$\Pr\left(h_u = \mathbf{sgn}\Big(\sum_{v \in V(u)} \mathbf{x}_v\Big)\right) \geq \frac{99}{100}.$$

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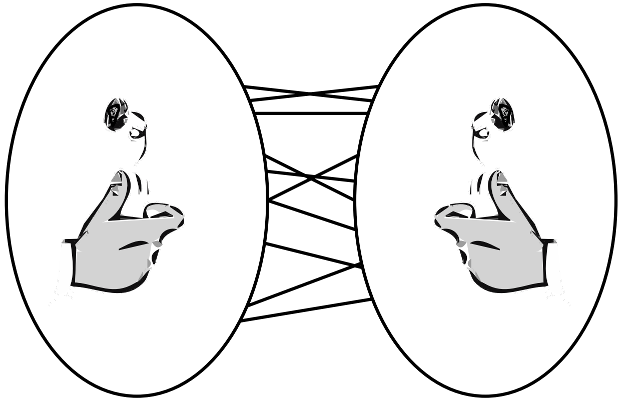

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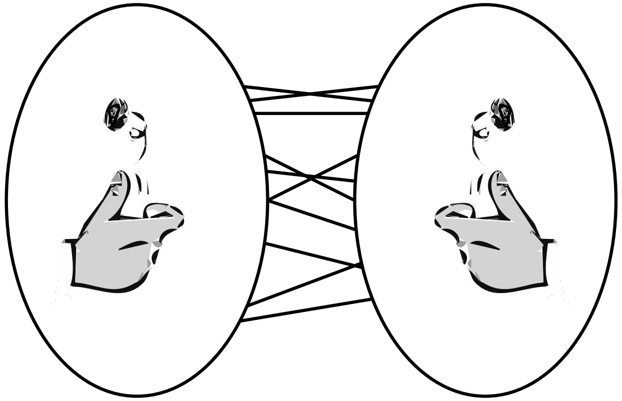

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sign of $\mathbf{x}_u$ at (local) time T

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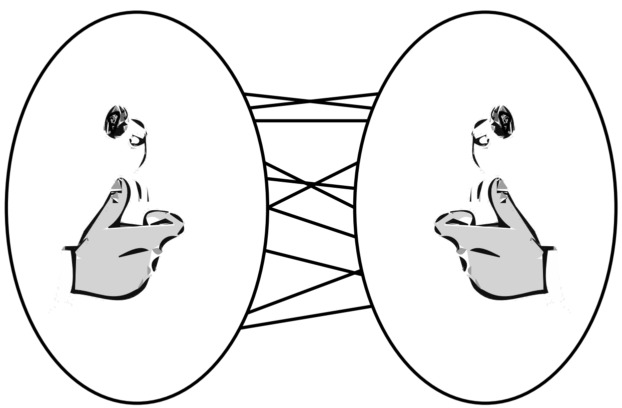

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Problem: bound $|U| = \#$*unlucky* nodes (i.e. $\mathbf{sgn}(\mathbf{x}_u^{(T)})$ is wrong with prob. $> 1/100$).

Analysis 2/4

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$\approx \epsilon^2 n$ nodes u are *bad*, namely

$$(\mathbf{x}_u^{(t)} - \sum_{v \in V(u)} \mathbf{x}_v^{(0)})^2 > \frac{\epsilon^2}{n}$$

then we can bound the *unlucky nodes* by bounding a *spreading process*:

- At time $10n \log n$, $\approx \epsilon^2 n$ nodes are *bad/unlucky*, and
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Analysis 2/4

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Next idea

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$$= \mathbf{E}\left[\|\mathbf{x}^{(t)} - \pi_{\mathbf{v}_{1,2}}(\mathbf{x}^{(0)})\|^2\right]$$

$$= \mathbf{E}\left[\|\pi_{\mathbf{v}_{\geq 2}}(\mathbf{x}_u^{(t)}) - \pi_{\mathbf{v}_2}(\mathbf{x}_u^{(0)})\|^2\right]$$

$$\leq \mathbf{E}\left[\|\prod P^{(i)}\pi_{\mathbf{v}_2}(\mathbf{x}_u^{(t)}) - \pi_{\mathbf{v}_2}(\mathbf{x}_u^{(0)})\|^2\right]$$

$$+ \mathbf{E}\left[\|\prod P^{(i)}\pi_{\mathbf{v}_{\geq 3}}(\mathbf{x}_u^{(0)})\|^2\right].$$

$\pi_{\mathbf{v}_i}(\mathbf{x})$ projection on i-th eigenspace

$P^{(i)}$ matrix of averaging at time i

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$$+\,\mathbf{E}\left[\|\prod P^{(i)}\pi_{\mathbf{v}_{\geq 3}}(\mathbf{x}_u^{(0)})\|^2\right]. \longleftarrow \text{Need double recurrence}$$

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Thank you!