On the Computational Power of Simple Dynamics

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informatik

Hamburg Universität, Hamburg - February 28, 2017

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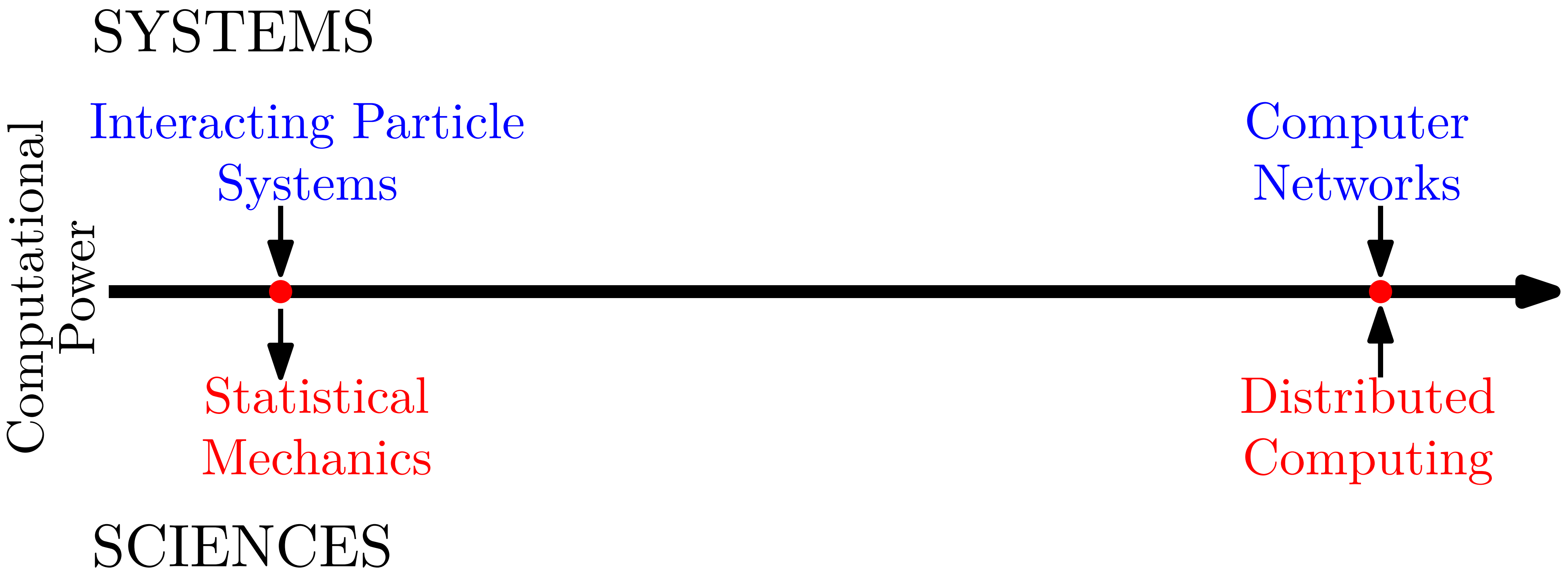
Communication in *Simple* Systems



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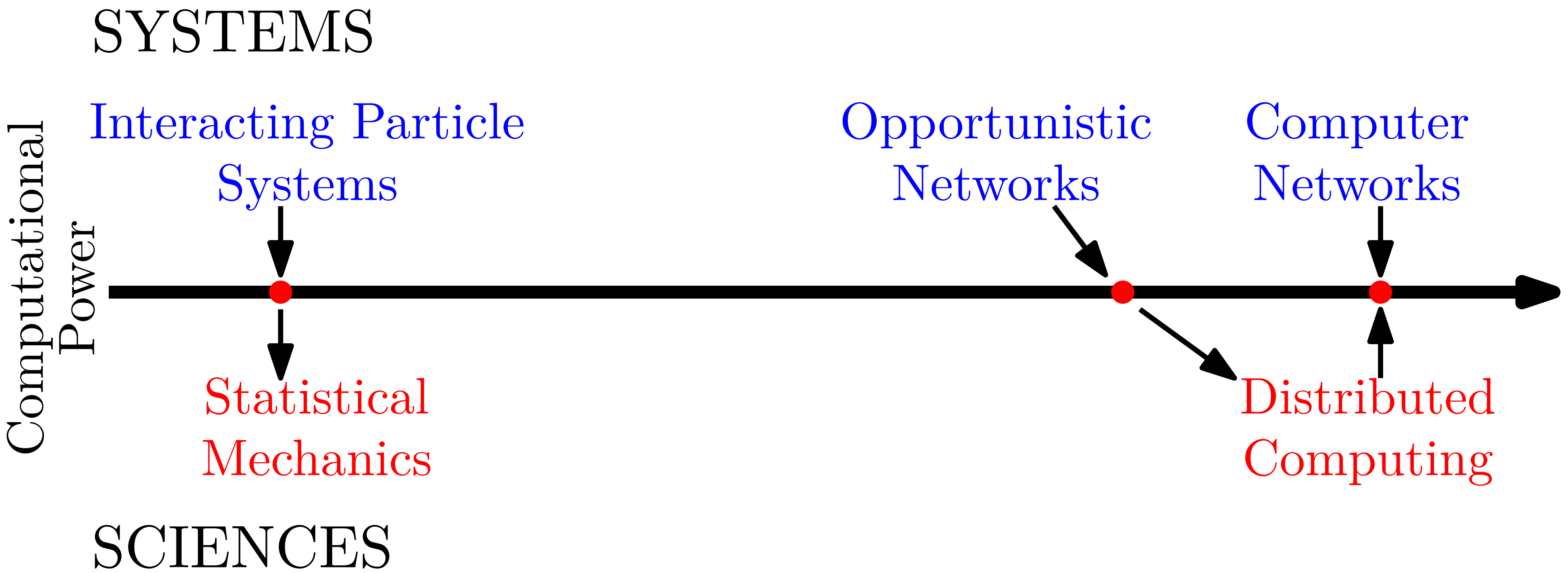

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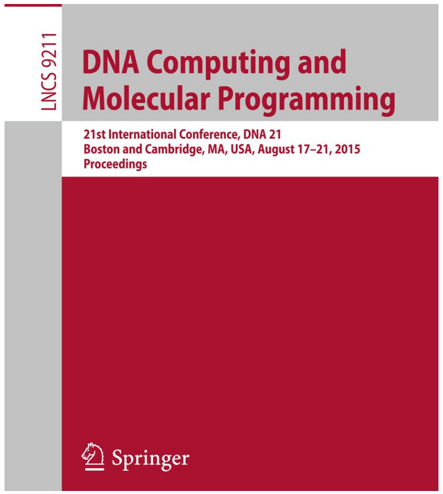
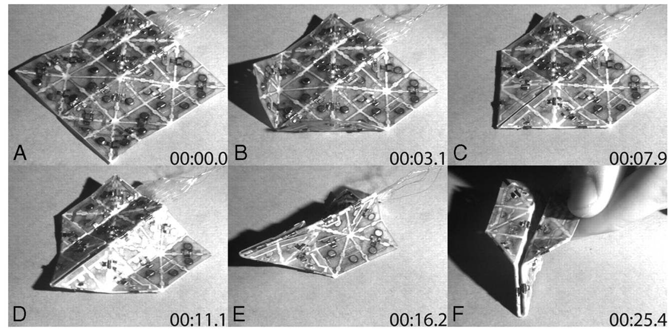
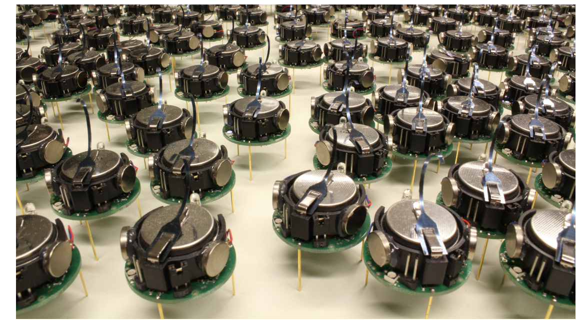
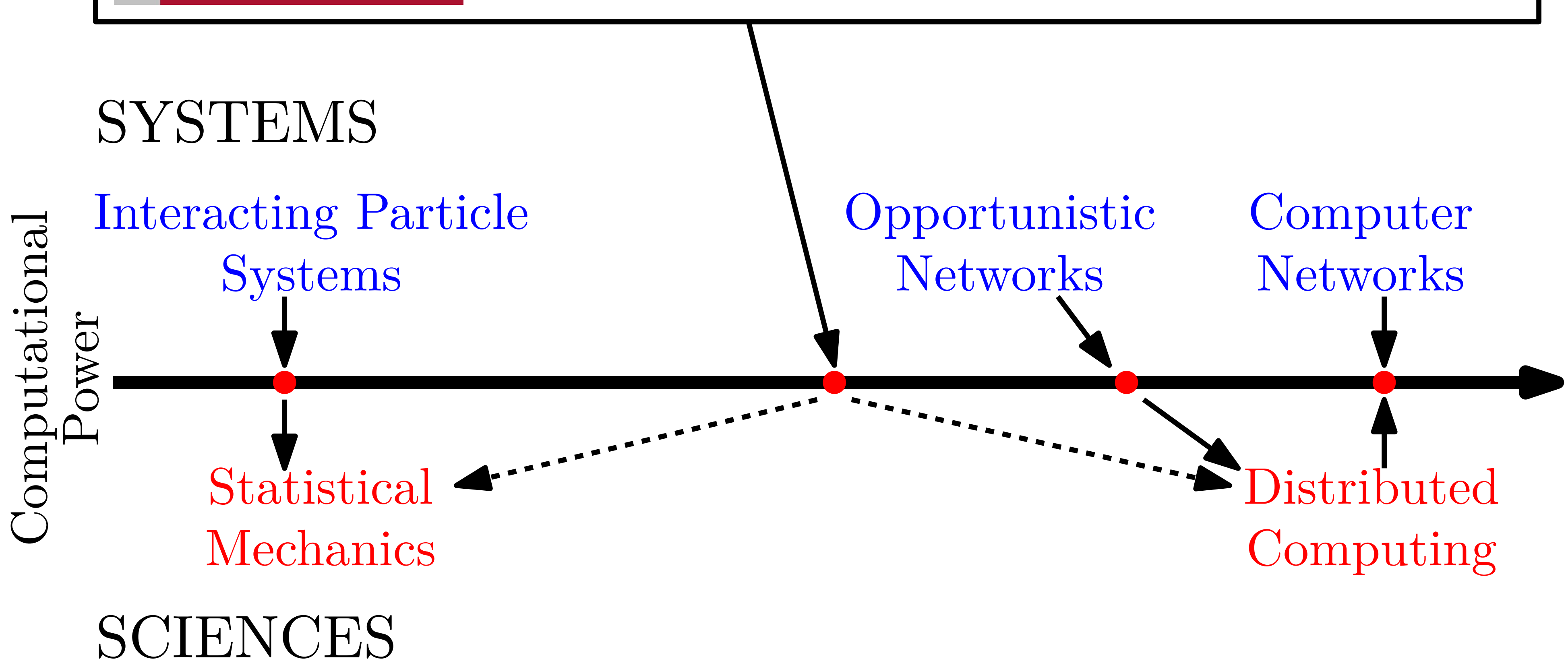






DNA/Molecular Computing, Programmable Matter, Swarms of Simple Robots



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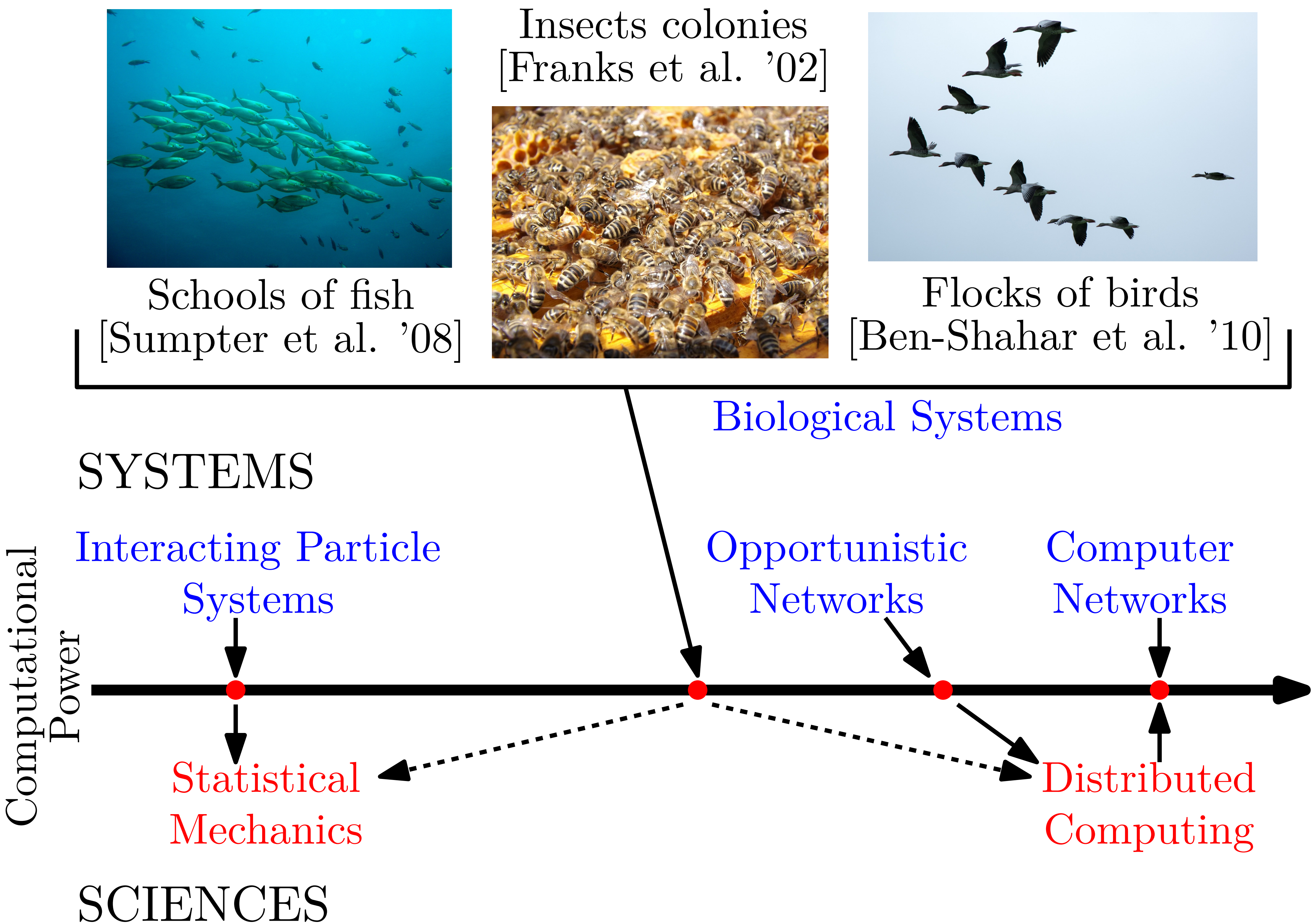

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Animal communication:

- Chaotic
- Anonymous
- Parsimonious
- Uni-directional (Passive)
- Noisy

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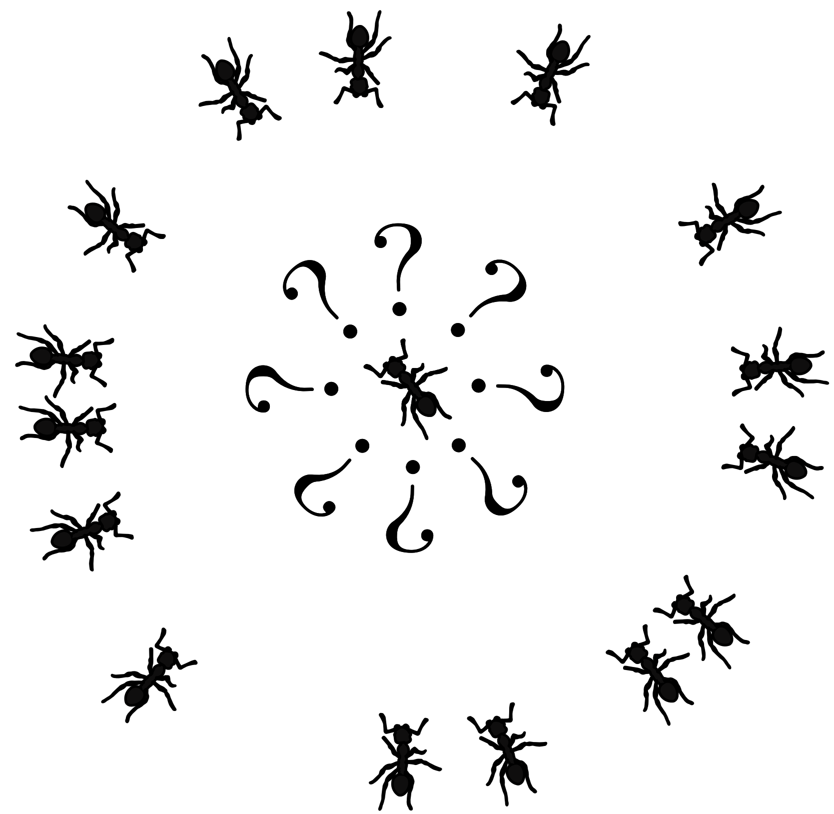
$\mathcal{PULL}(h, \ell)$ **model [Demers '88]**: at each round each agent can *observe* h other agents chosen independently and uniformly at random, and *shows* ℓ *bits* to her observers.



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(informal) *Very simple* distributed algorithms: For every graph, agent and round, states are updated according to fixed rule of current state and symmetric function of states of neighbors.

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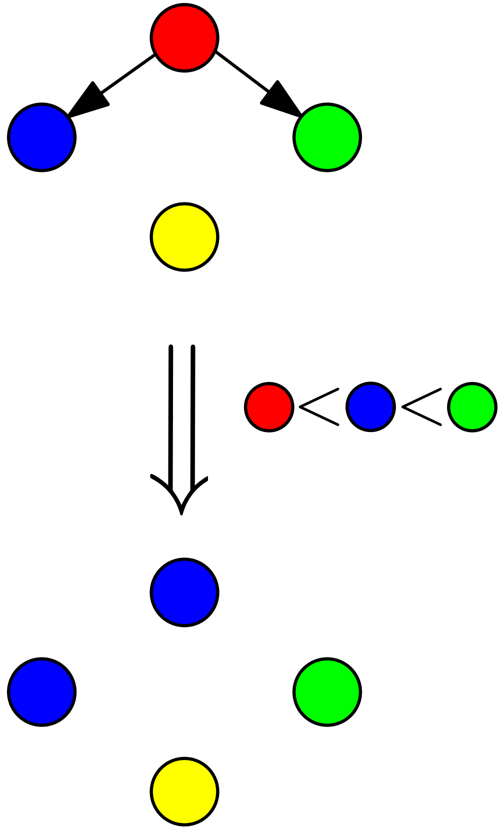
Examples of Dynamics

- 3-Median dynamics



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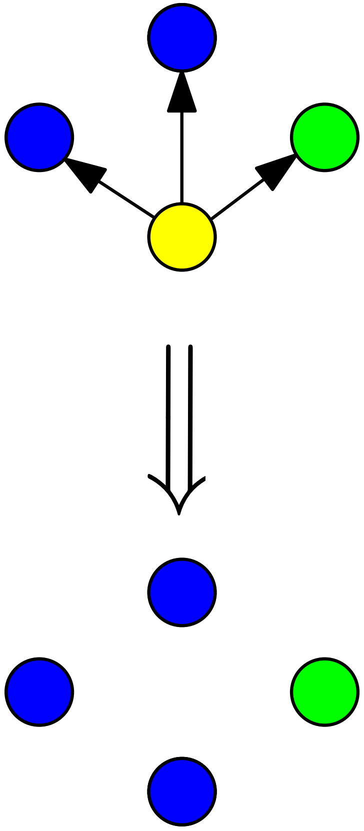

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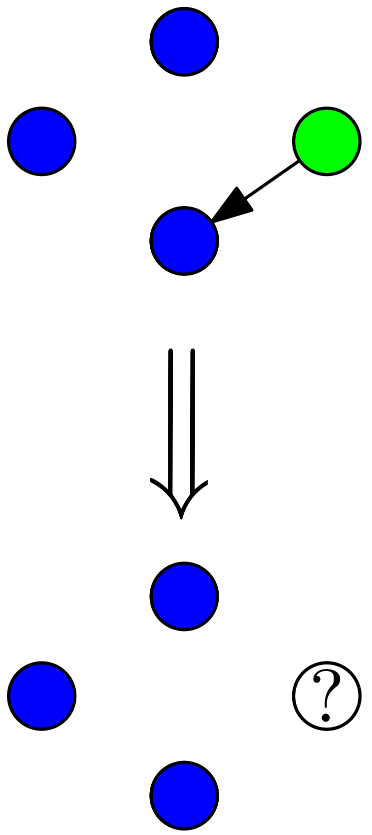

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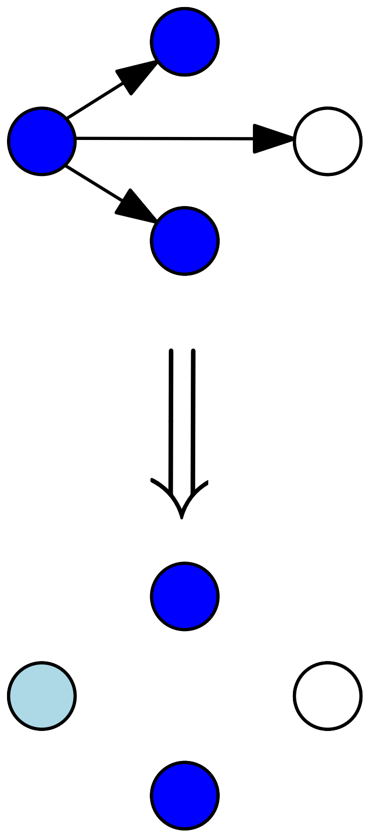
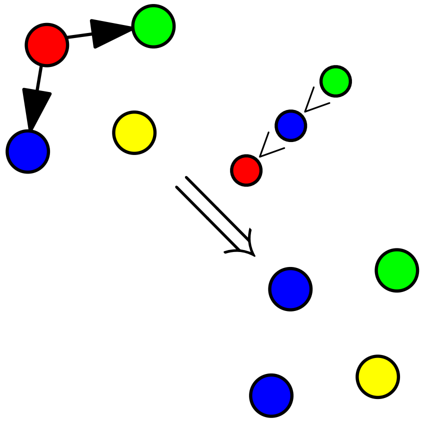


The Power of Dynamics: Plurality Consensus

Computing the Median



- 3-Median dynamics [Doerr et al. '11]. Converge to $\mathcal{O}(\sqrt{n \log n})$ approximation of median of system in $\mathcal{O}(\log n)$ rounds w.h.p., even if $\mathcal{O}(\sqrt{n})$ states are arbitrarily changed at each round ($\mathcal{O}(\sqrt{n})$-bounded adversary).

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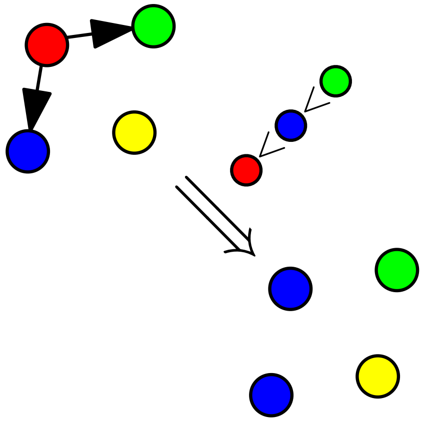

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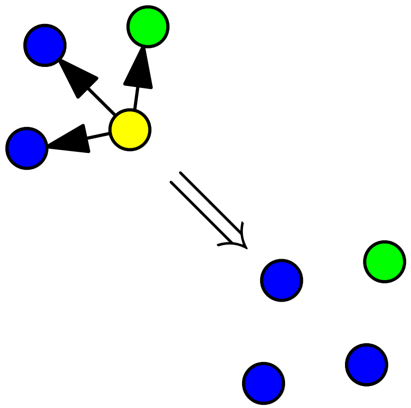

- 3-Majority dynamics [SPAA '14, SODA '16]. If plurality has **bias** $\mathcal{O}(\sqrt{kn \log n})$, converges to it in $\mathcal{O}(k \log n)$ rounds w.h.p., even against $o(\sqrt{n/k})$-bounded adversary. Without bias, converges in $\text{poly}(k)$. h-majority converges in $\Omega(k/h^2)$.

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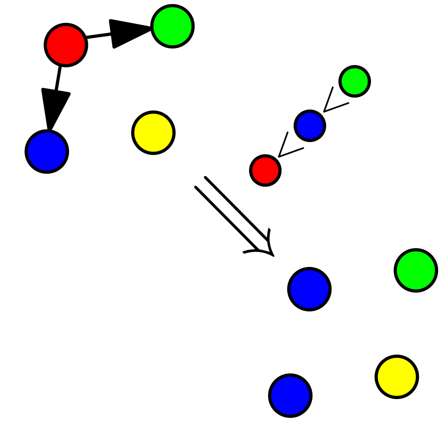

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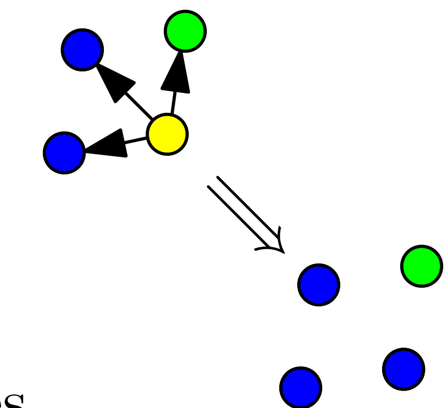

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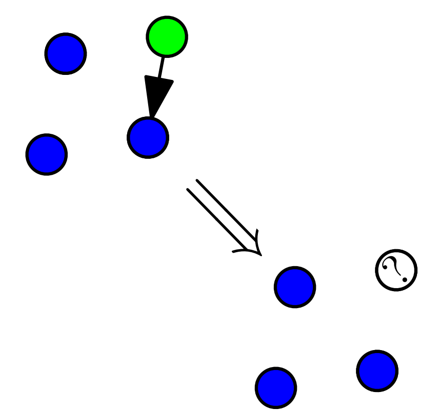
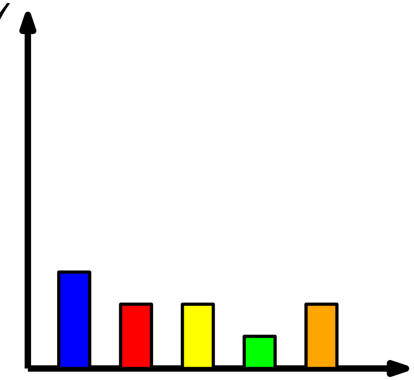


- Undecided-State dynamics [SODA '15]. If majority/second-majority ($c_{maj}/c_{2^{nd}maj}$) is at least $1+\epsilon$, system converges to plurality within $\tilde{\Theta}(\text{md}(\mathbf{c}))$ rounds w.h.p.

A Global Measure of Bias

$\mathrm{md}(\mathbf{c}^{(\mathbf{0})}) := \sum_{i=1}^{k} \left(\frac{c_i^{(0)}}{c_{maj}^{(0)}}\right)^2 = 1 + \mathcal{D}\Big($



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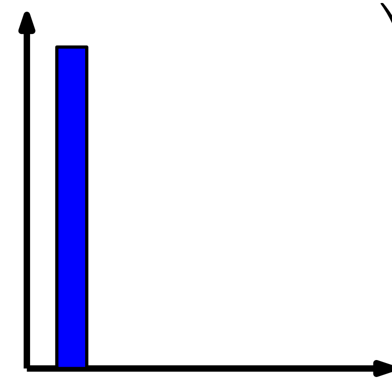
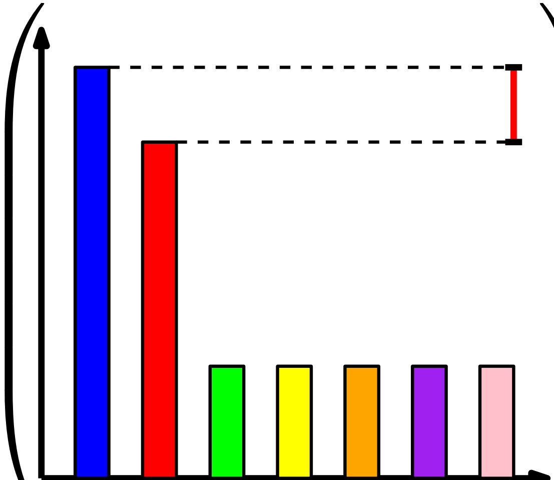


$\Big)$

$1 \leq \mathrm{md}\Big($



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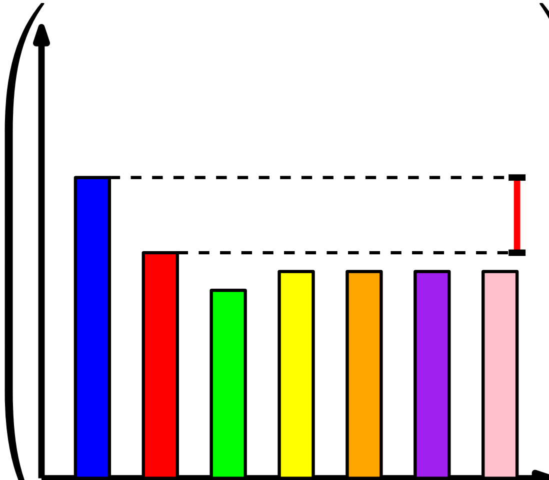

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The Median, the Mode and... the Mean

Dynamics can solve Consensus, Median, Majority, in robust and fault tolerant ways, but this is trivial in centralized setting.

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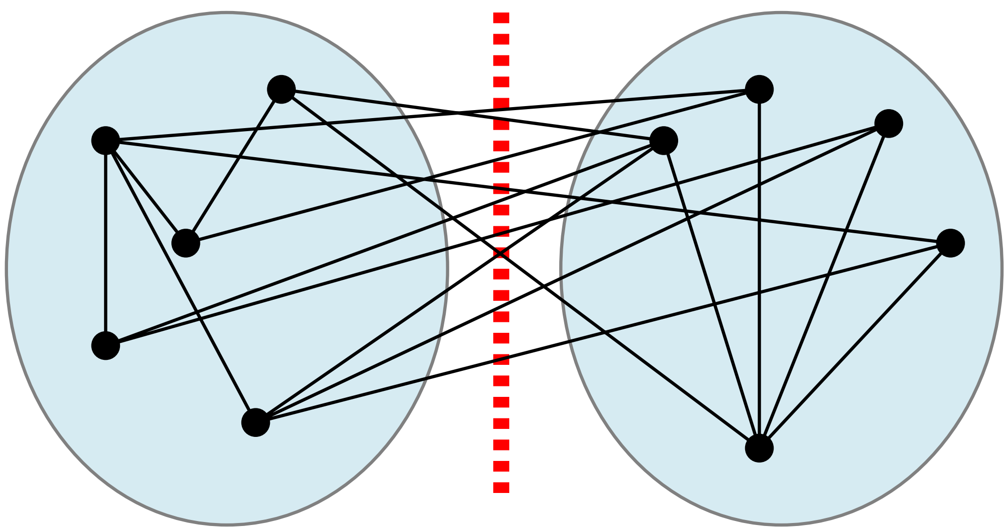
Can dynamics solve a problem non-trivial in centralized setting?

Community Detection as Minimum Bisection

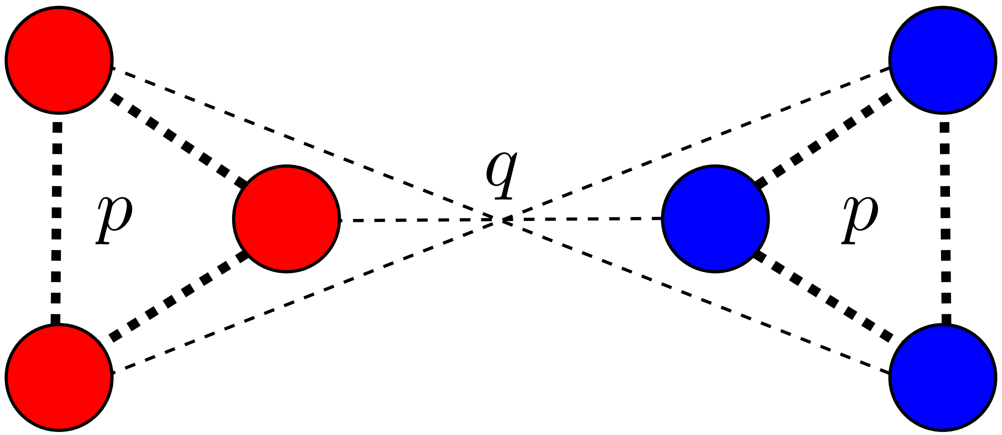
Minimum Bisection Problem.
Input: a graph G with $2n$ nodes.
Output: $S = \arg\min_{\substack{S \subset V \\ |S| = n}} E(S, V - S)$.



[Garey, Johnson, Stockmeyer '76]:
Min-Bisection is *NP-Complete*.

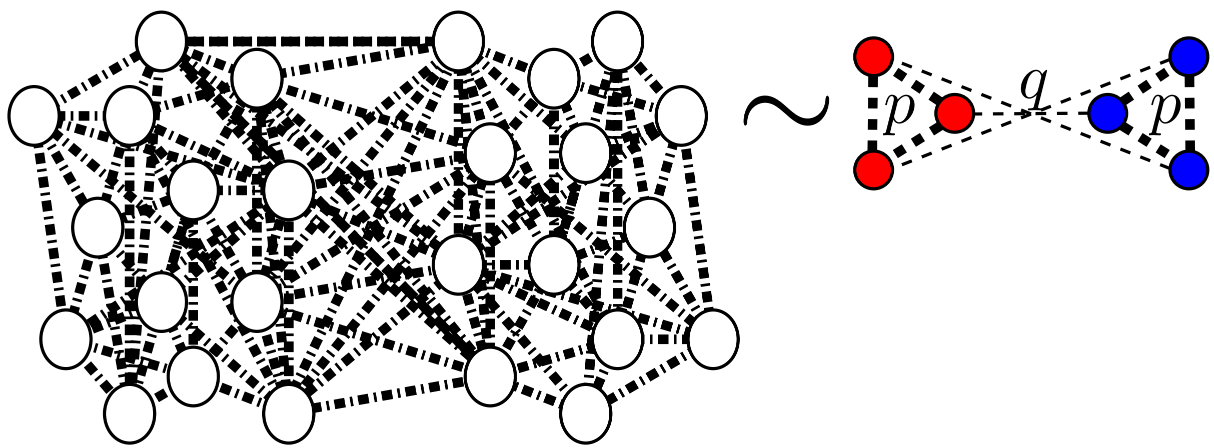
The Stochastic Block Model

Stochastic Block Model (SBM). Two "communities" of equal size V_1 and V_2, each edge inside a community included with probability $p = \frac{a}{n}$, each edge across communities included with probability $q = \frac{b}{n} < p$.



The Stochastic Block Model

Reconstruction problem. Given graph generated by SBM, find original partition.



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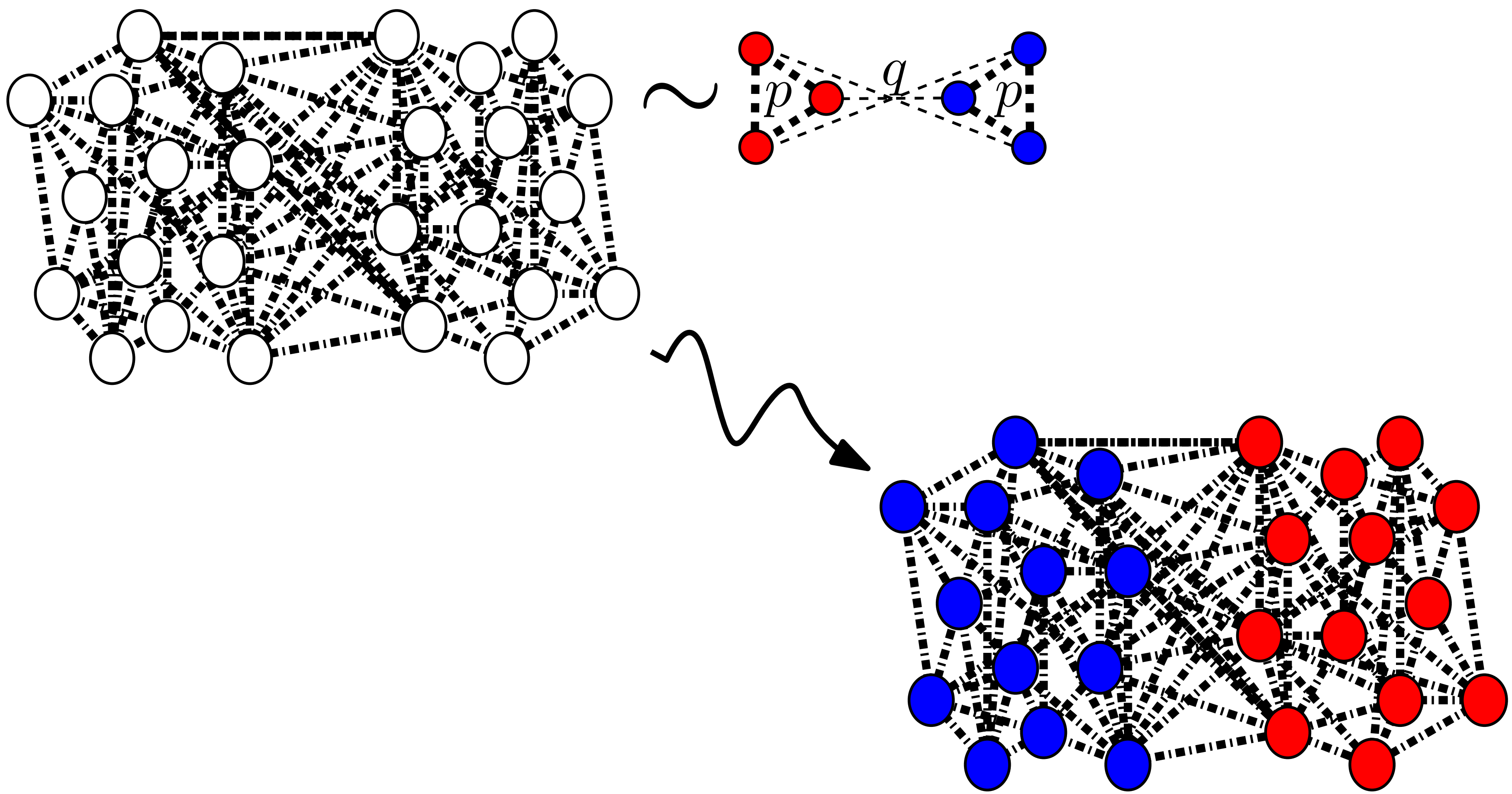
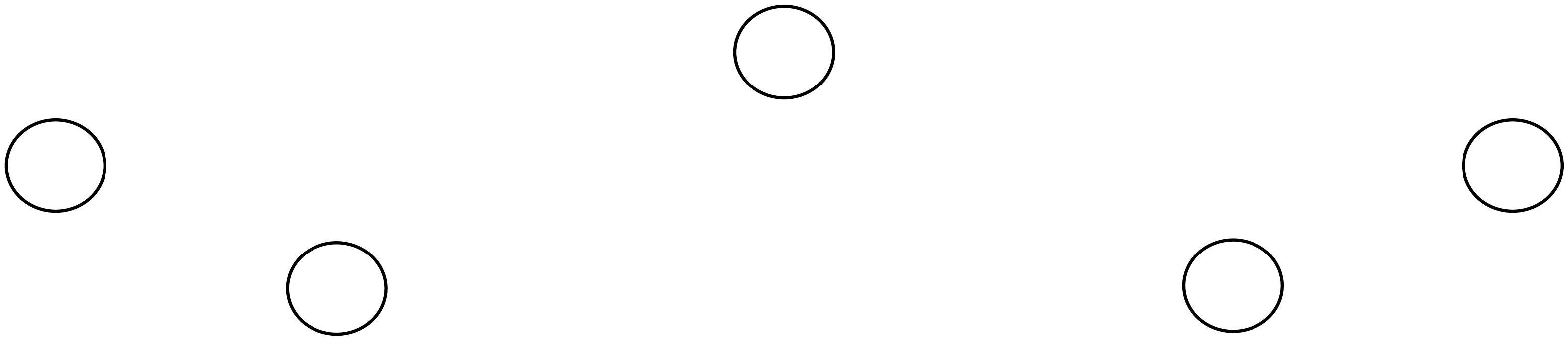


The Averaging Dynamics

Al nodes at the same time:

- At $t = 0$, randomly pick value $x^{(t)} \in \{\mathbf{+1}, \mathbf{-1}\}$.
- Then, at each round
  1. Set value $x^{(t)}$ to average of neighbors,
  2. Set label to **blue** if $x^{(t)} < x^{(t-1)}$, **red** otherwise.



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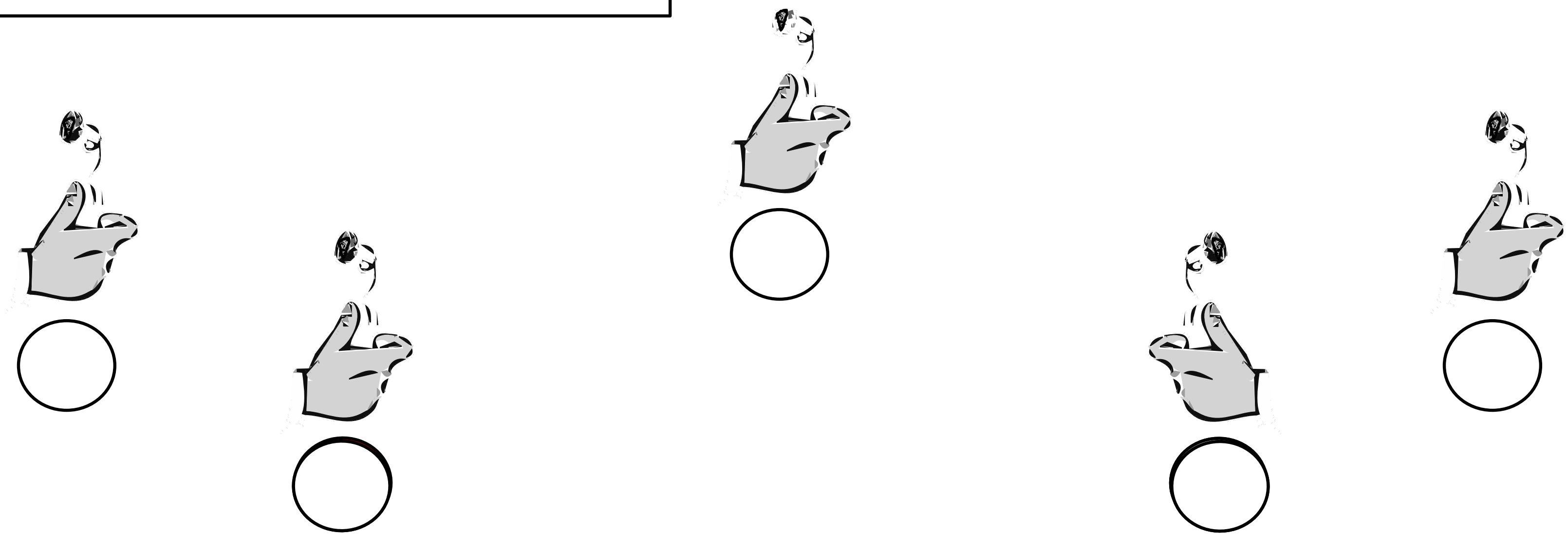

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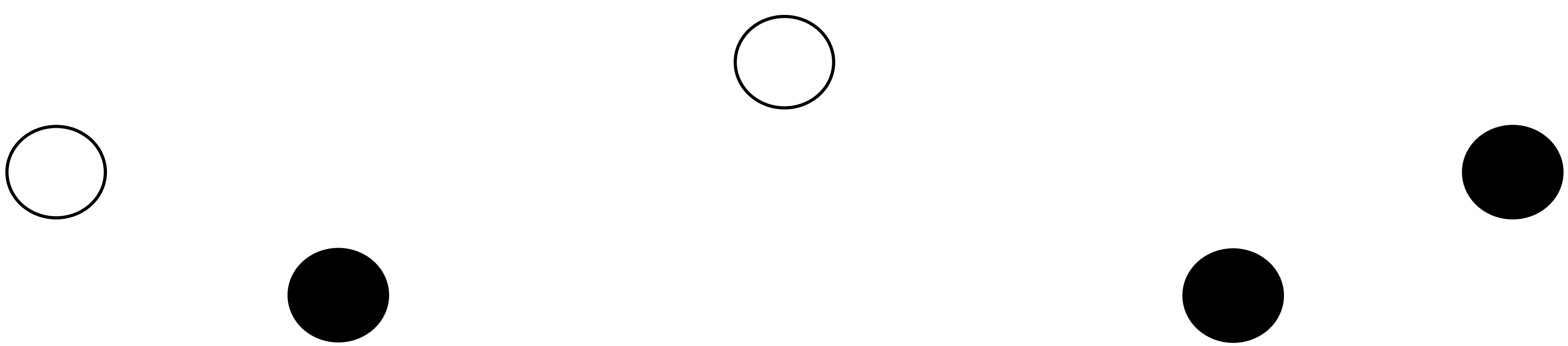

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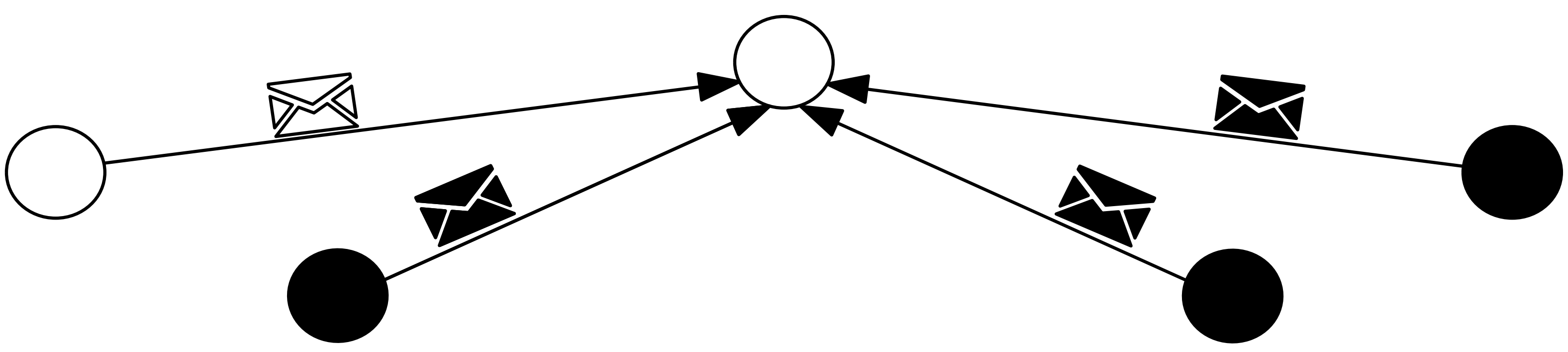

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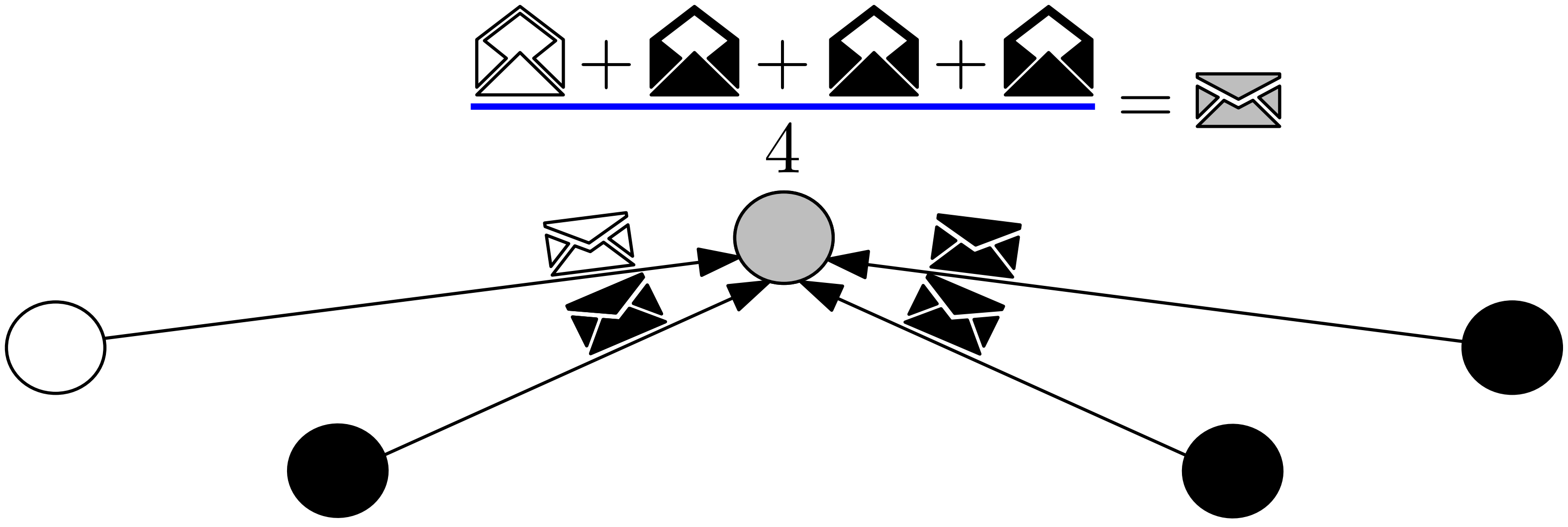

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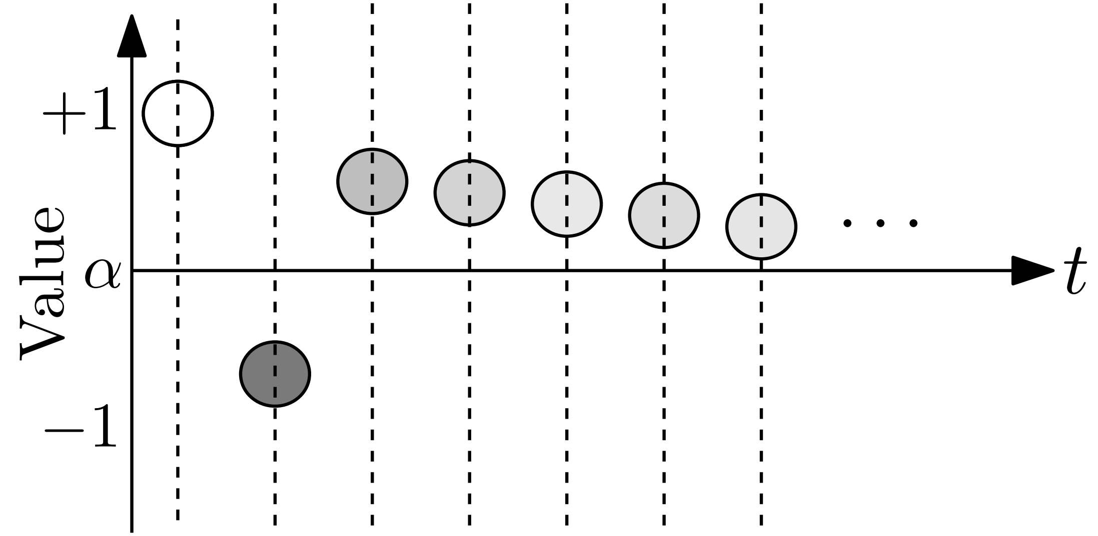

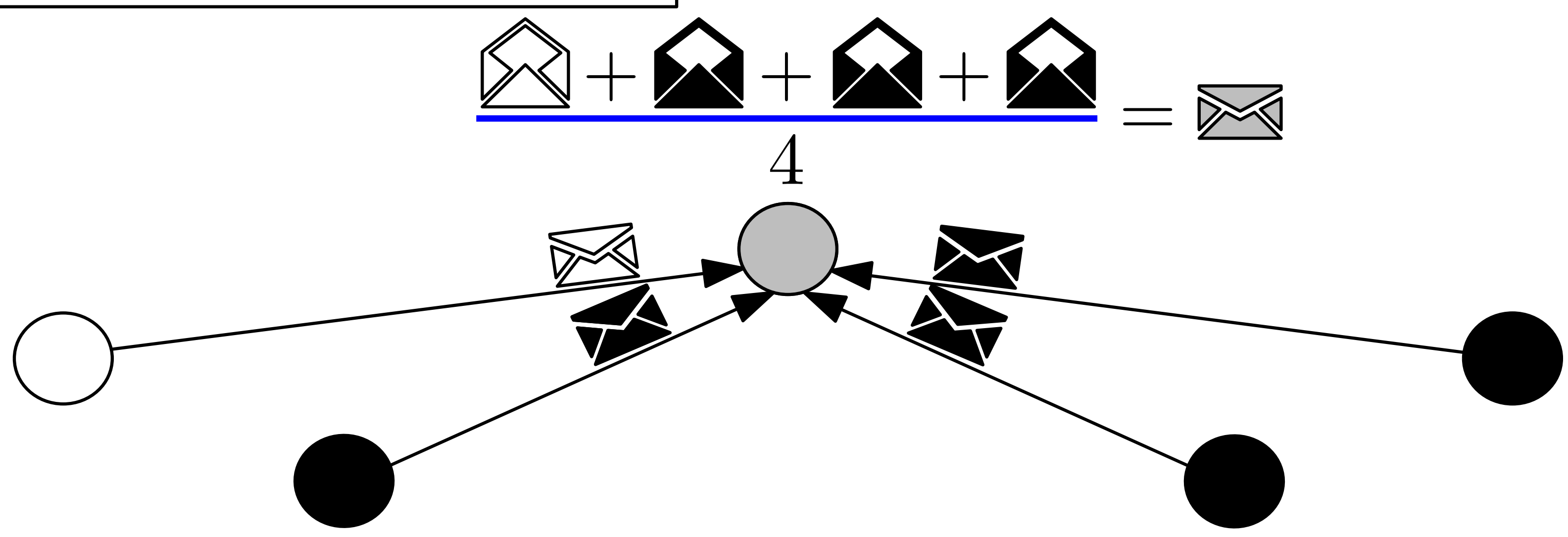

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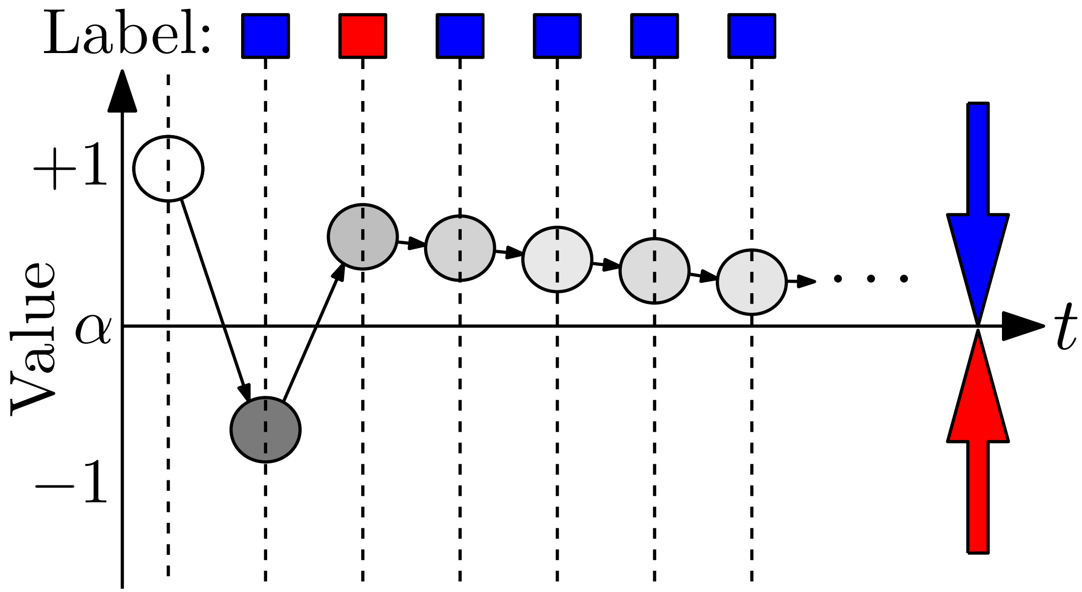

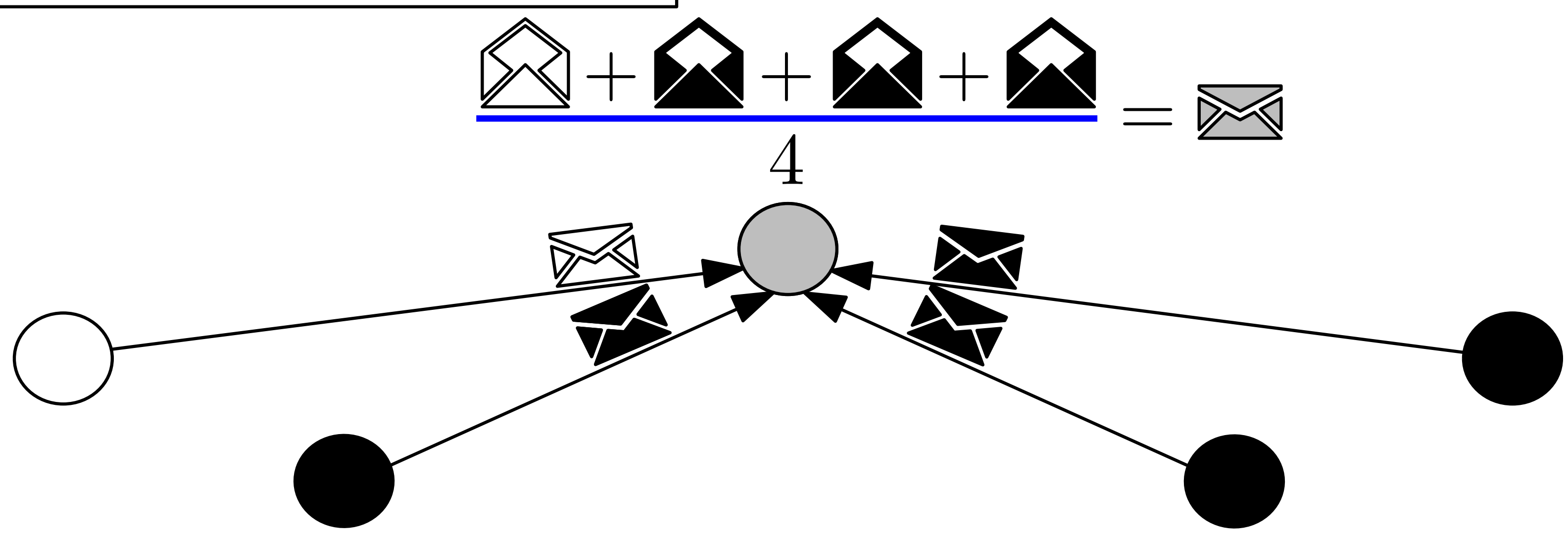

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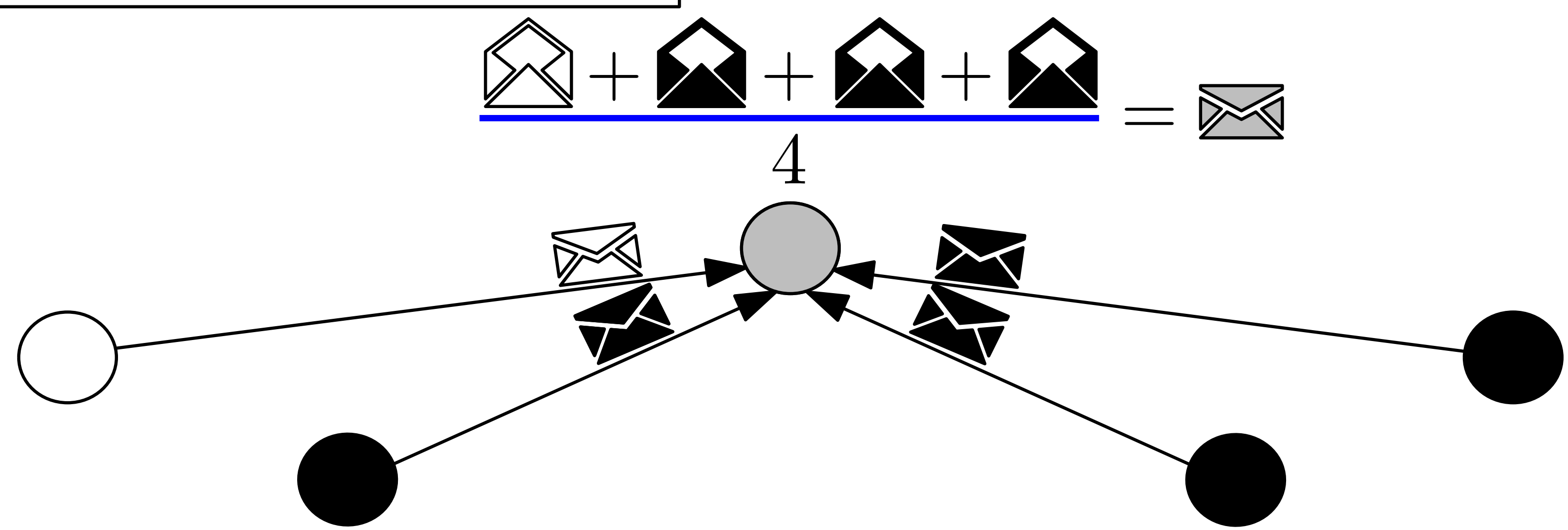
Well studied process [Shah '09]:

- Converges to (weighted) global average of initial values,
- Convergence time = mixing time of G,
- Important applications in fault-tolerant self-stabilizing consensus.



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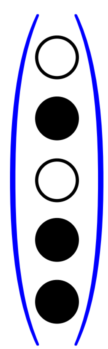
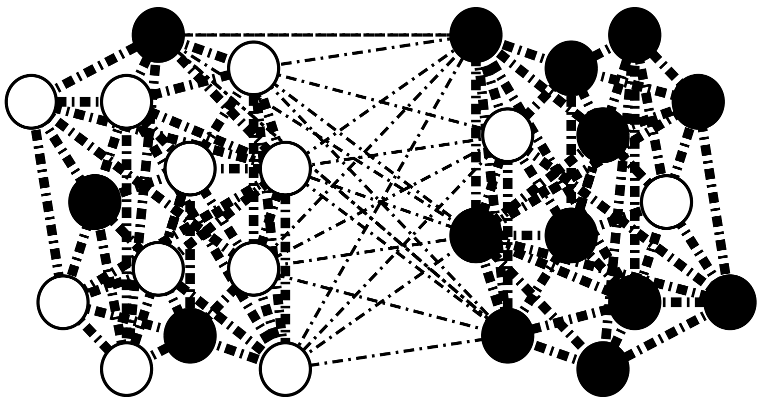
Averaging is a **linear** dynamics

$\mathbf{x}^{(t)} =$ 

$$\mathbf{x}^{(t)} = P \cdot \mathbf{x}^{(t-1)} = P^t \cdot \mathbf{x}^{(0)}$$

P transition matrix of random walk

Community Detection via Averaging Dynamics



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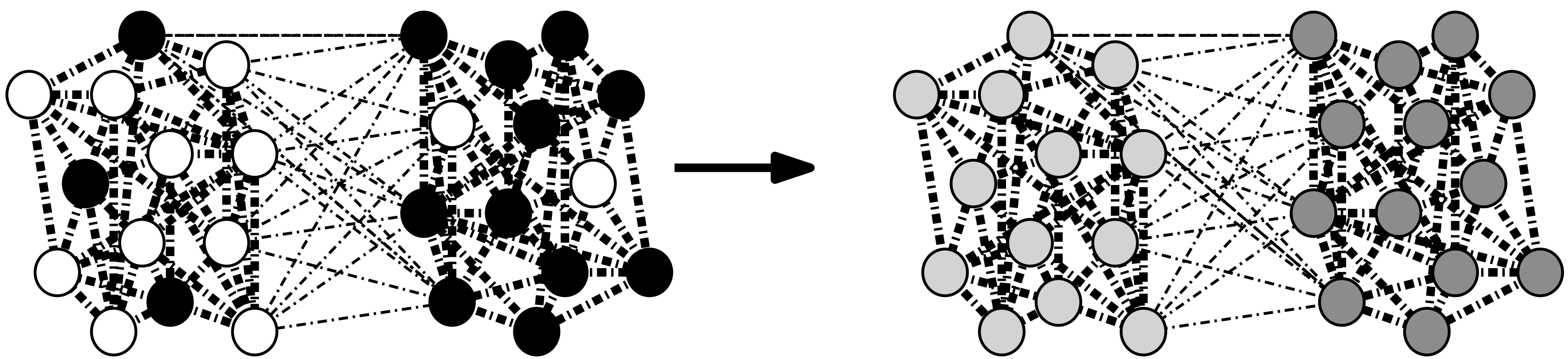

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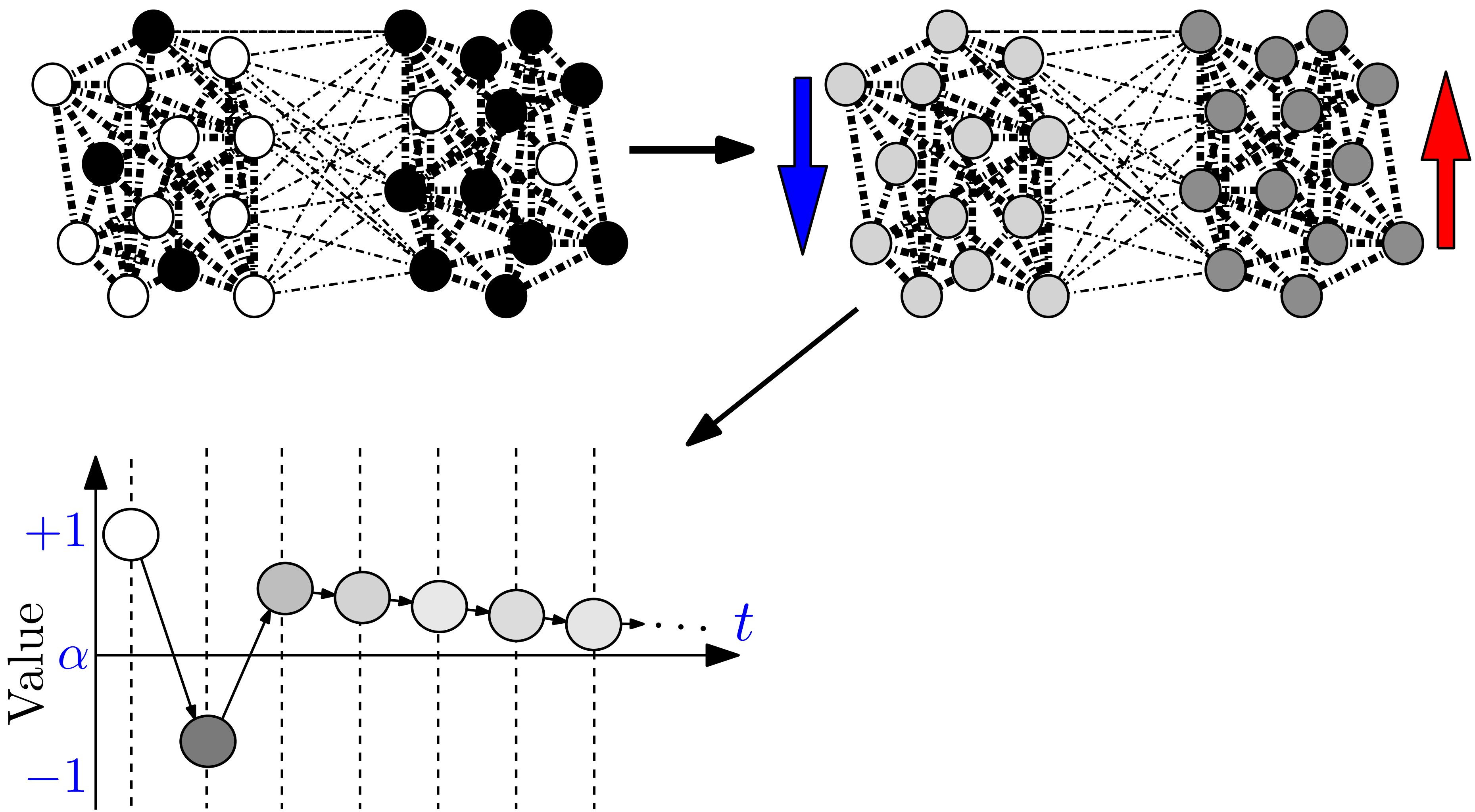

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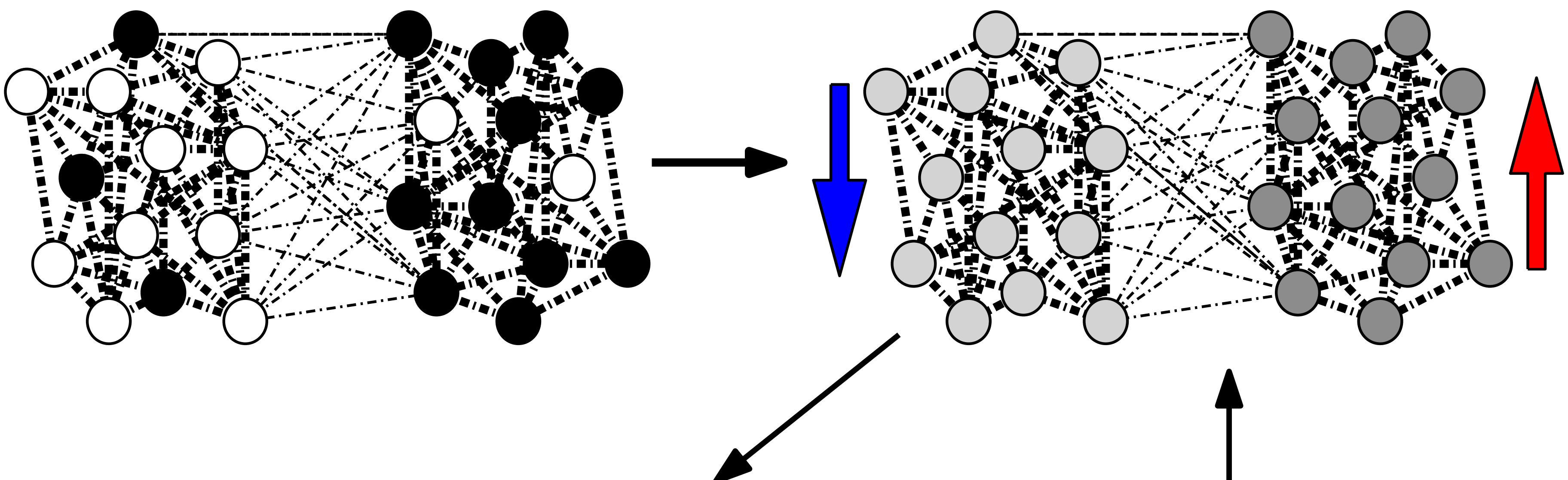
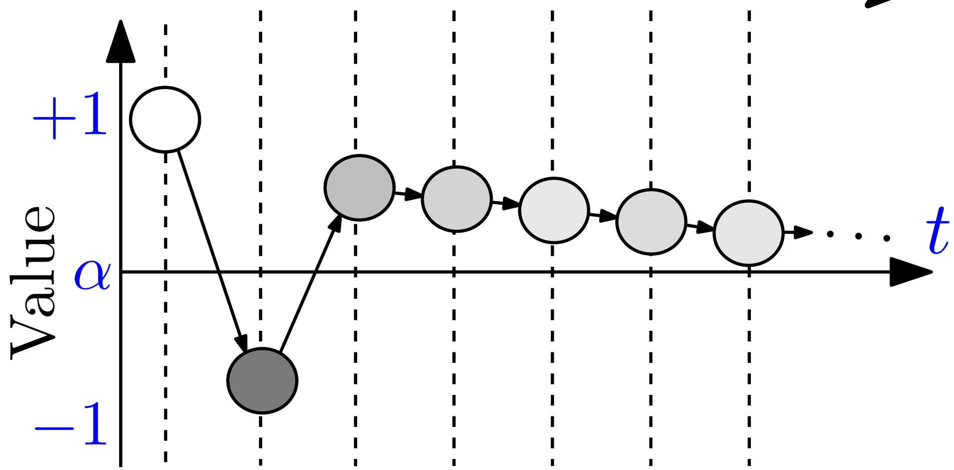




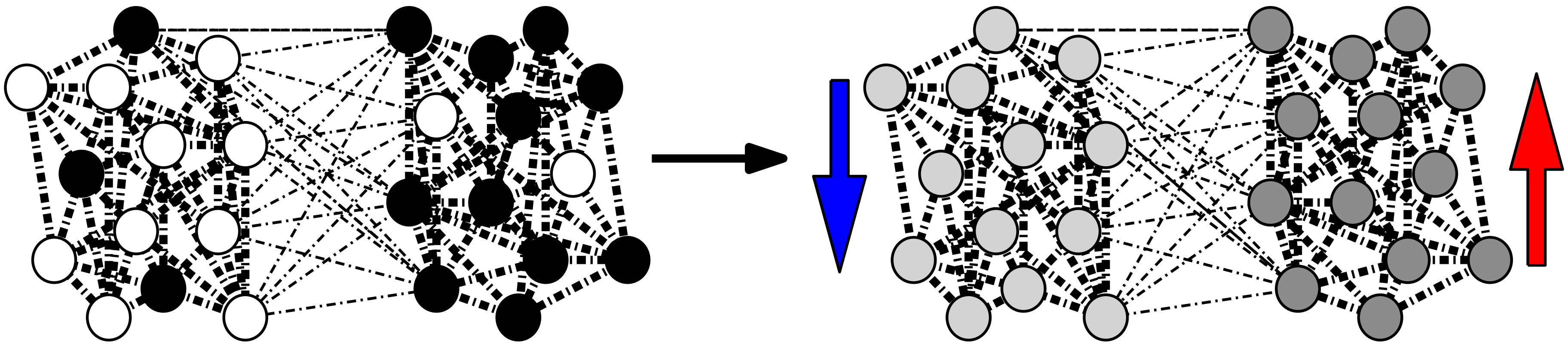
$v_1, \ldots, v_n$ eigenvectors of random walk matrix P:

$v_1 = \mathbb{1} = (1, \ldots, 1)$

$v_2 \approx \chi = (1, \ldots, 1, -1, \ldots, -1)$

"nice" graph

Community Detection via Averaging Dynamics



[SODA '17] (Informal). $G = (V_1 \dot{\bigcup} V_2, E)$ s.t.
i) $\chi = \mathbf{1}_{V_1} - \mathbf{1}_{V_2}$ close to right-eigenvector of eigenvalue λ_2 of transition matrix of G, and
ii) gap between λ_2 and $\lambda = \max\{\lambda_3, |\lambda_n|\}$ sufficiently large,
then Averaging (approximately) identifies (V_1, V_2).

Thank you!