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Chrstim🎄s Lecture

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max planck institut
informatik

December 2017

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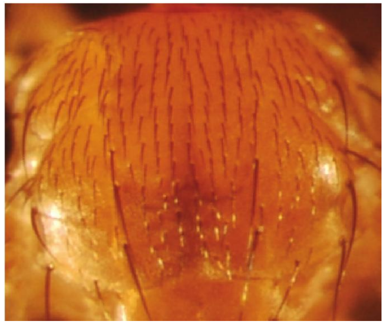

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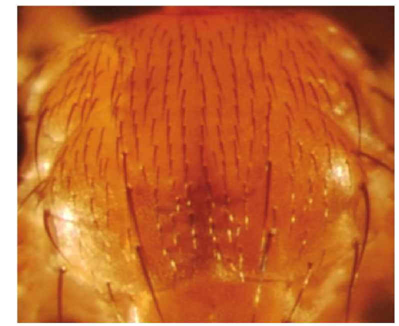

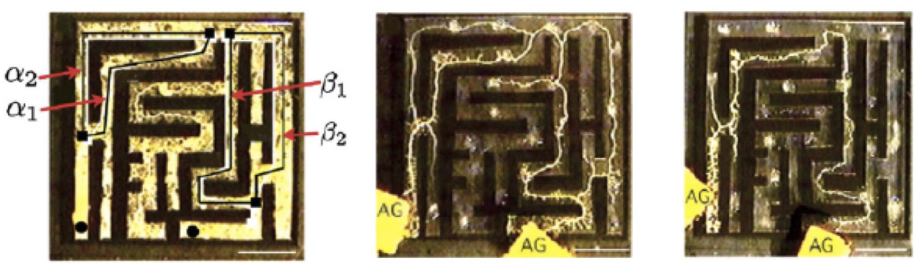

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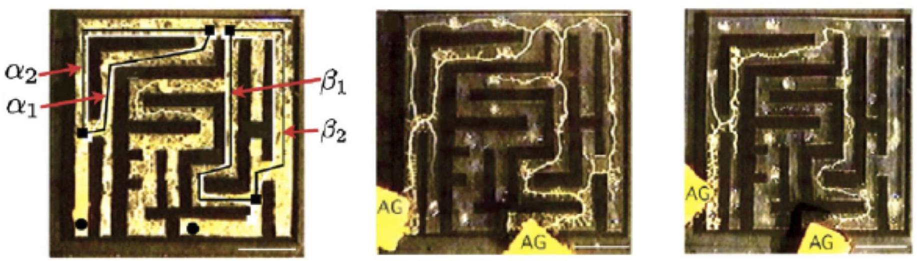

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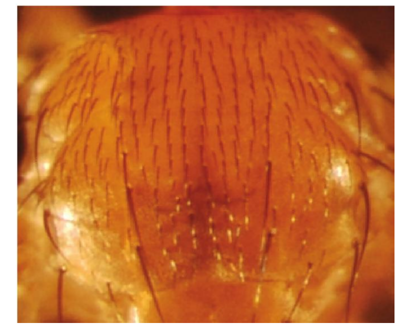
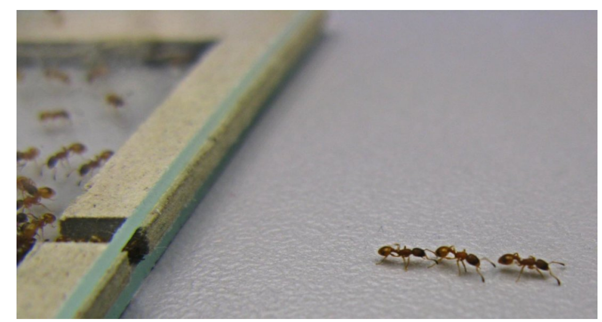


How do ants decide where to relocate their nest? [GMRL '15]



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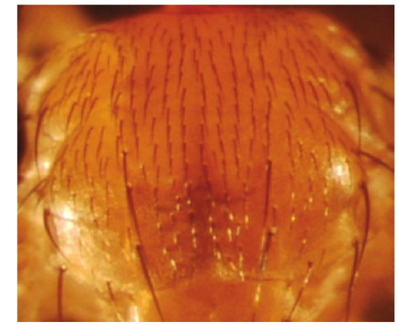

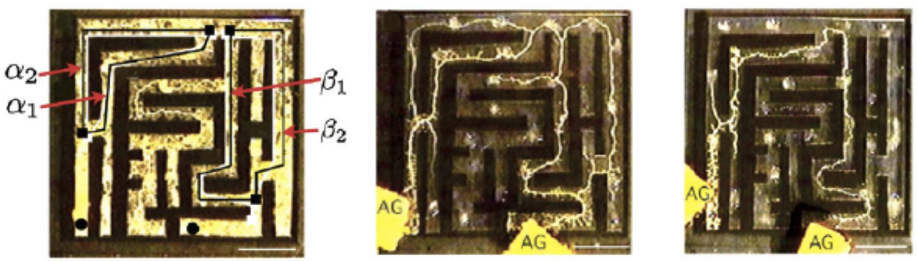

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How ants perform collective navigattion [FHBGKKF '16]

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References

- **[Chazelle '09]** B. Chazelle, "Natural algorithms," in Proceedings of the twentieth Annual ACM-SIAM Symposium on Discrete Algorithms, 2009, pp. 422–431.
- **[AABHBB '11]** Y. Afek, N. Alon, O. Barad, E. Hornstein, N. Barkai, and Z. Bar-Joseph, "A biological solution to a fundamental distributed computing problem," Science, vol. 331, no. 6014, pp. 183–185, Jan. 2011.
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- **[GMRL '15]** M. Ghaffari, C. Musco, T. Radeva, and N. Lynch, "Distributed House-Hunting in Ant Colonies," in Proceedings of the 2015 ACM Symposium on Principles of Distributed Computing, New York, NY, USA, 2015, pp. 57–66.
- **[FHBGKKF '16]** E. Fonio et al., "A locally-blazed ant trail achieves efficient collective navigation despite limited information," eLife, vol. 5, p. e20185, Nov. 2016.
- **A survey:** S. Navlakha and Z. Bar-Joseph, "Distributed information processing in biological and computational systems," Communications of the ACM, vol. 58, no. 1, pp. 94–102, Dec. 2014.

Ants are Cool!

Ants invented **agricolture**

The ants cultivate species of fungi, feeding them with freshly cut plant material and keeping them free from pests and molds, and if they notice that a type of leaf is toxic to the fungus, they will no longer collect it. Some of this fungi no longer produce spores: they fully domesticated their fungal partner 15 million years ago.

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Some colonies conduct ritualized tournaments: Opposing colonies summon their worker forces to the tournament area, where hundreds of ants perform highly stereotyped display of fights. When one colony is considerably stronger than the other, the tournaments end quickly and the weaker colony is sacked. - The Ants, B. Hölldobler and E. O. Wilson

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Ants invented **slavery**

Slave-making ants are brood parasites that capture broods of other ant species to increase the worker force of their colony. After emerging in the slave-maker nest, slave workers work as if they were in their own colony, while parasite workers only concentrate on replenishing the labor force from neighboring host nests.

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Ants are Cool!

Ants invented **architectures**

When army ants need to cross a large gap, they simply build a bridge - with their own bodies. Linking together, the ants can move their living bridge from its original point, allowing them to cross gaps and create shortcuts across rainforests in Central and South America.

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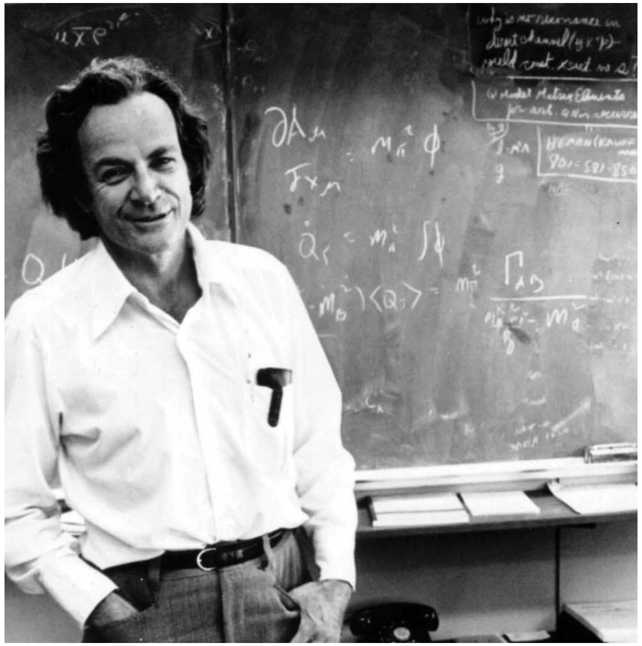

Ants puzzled **Feynman**

One question that I wondered about was why the ant trails look so straight and nice. The ants look as if they know what they're doing, as if they have a good sense of geometry. Yet the experiments that I did to try to demonstrate their sense of geometry didn't work. Many years later, when I was at Caltech . . .

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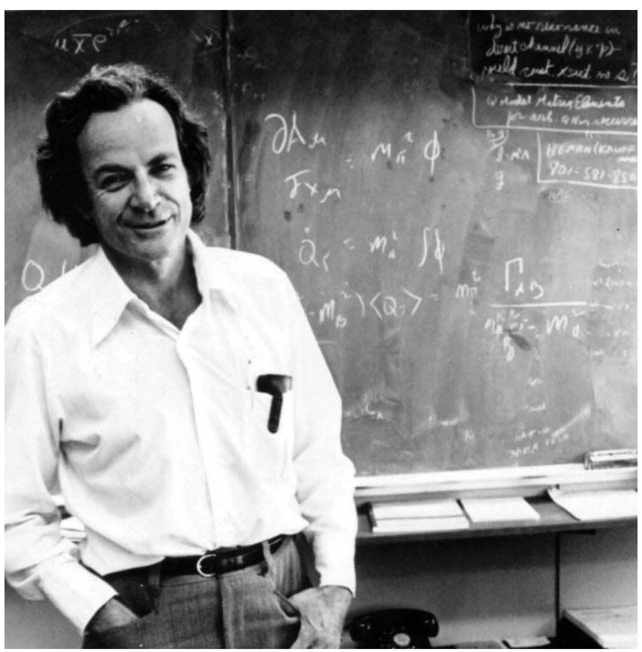

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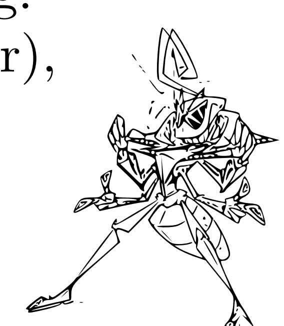
And **more...**

Have a look at the many books (e.g. Hölldobler), or just Youtube.



Ants are symbol of *Biological Distributed Algorithm*

Lots of work going on:

- T. Radeva, "A Symbiotic Perspective on Distributed Algorithms and Social Insects," PhD Thesis, MIT, 2017.
- A. Cornejo, A. Dornhaus, N. Lynch, and R. Nagpal, "Task Allocation in Ant Colonies," in Distributed Computing, Springer, 2014, pp. 46–60.
- Y. Afek, R. Kecher, and M. Sulamy, "Faster task allocation by idle ants," arXiv:1506.07118 [cs], Jun. 2015.
- Y. Emek, T. Langner, D. Stolz, J. Uitto, and R. Wattenhofer, "How Many Ants Does It Take to Find the Food?," in SIROCCO, Springer , 2014, pp. 263–278.
- O. Feinerman and A. Korman, "The ANTS problem," Distrib. Comput., pp. 1–20, Oct. 2016. (an entire myrmecology lab!)
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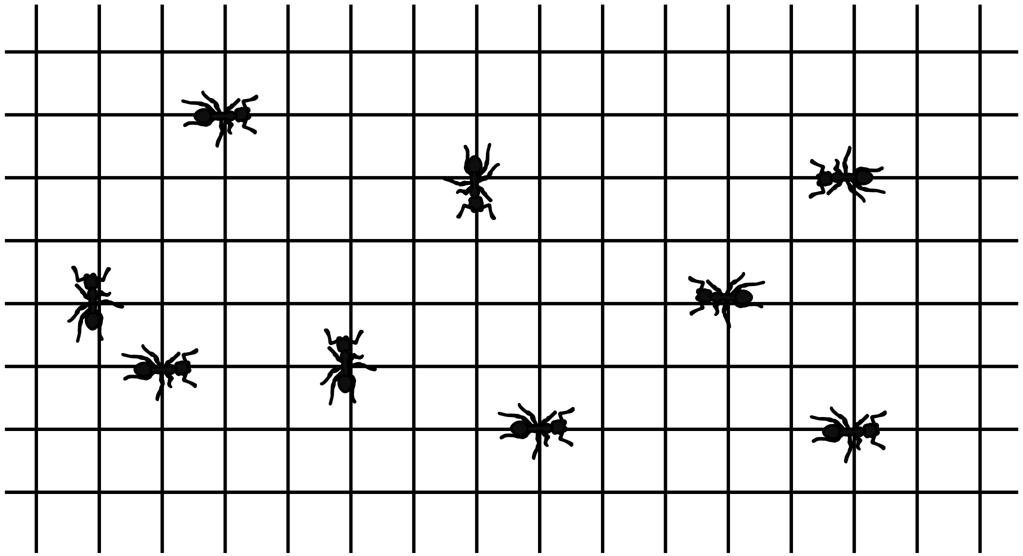
Today we talk about

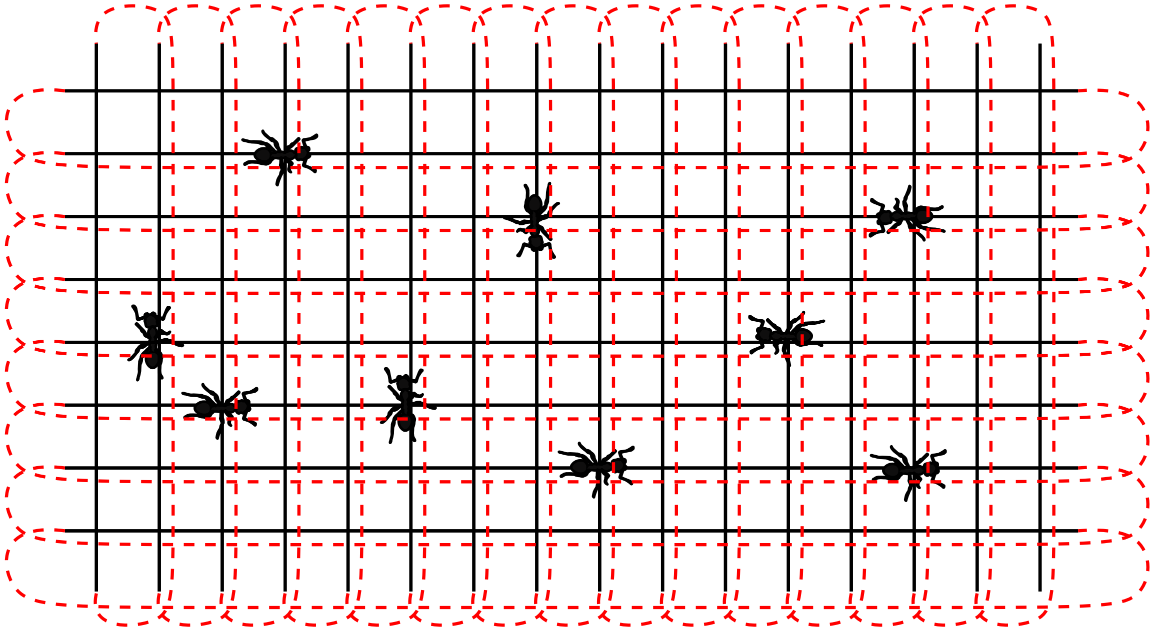
- C. Musco, H.-H. Su, and N. Lynch, "Ant-Inspired Density Estimation via Random Walks: Extended Abstract," In PODC 2016, pp. 469–478.
- C. Musco, H.-H. Su, and N. A. Lynch, "Ant-inspired density estimation via random walks," In PNAS, vol. 114, no. 40, pp. 10534–10541, Oct. 2017.

Density Estimation Problem



A graph (say a grid) of size $\sqrt{A} \times \sqrt{A}$, n ants. Each ant wants to learn the density $d = n/A$.

Density Estimation Problem



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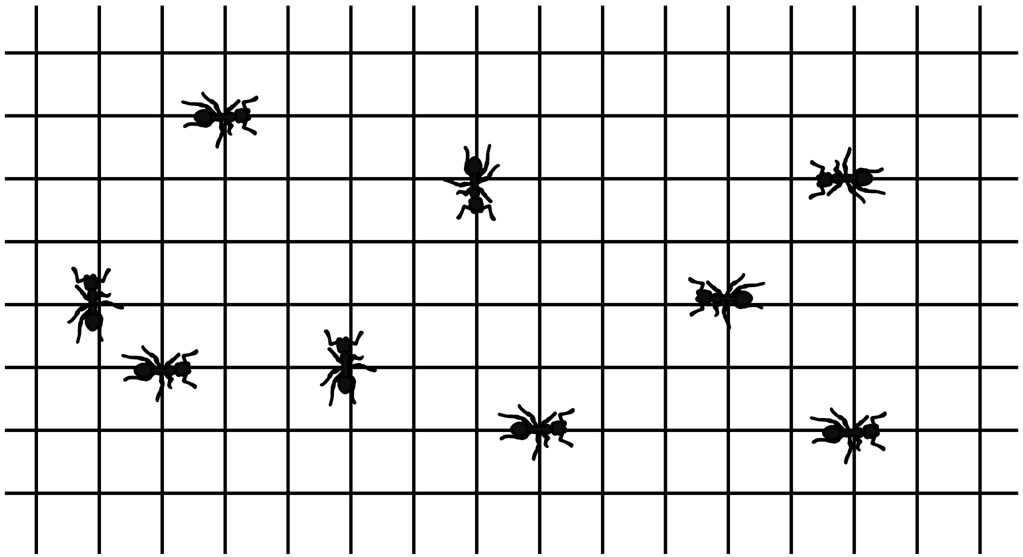

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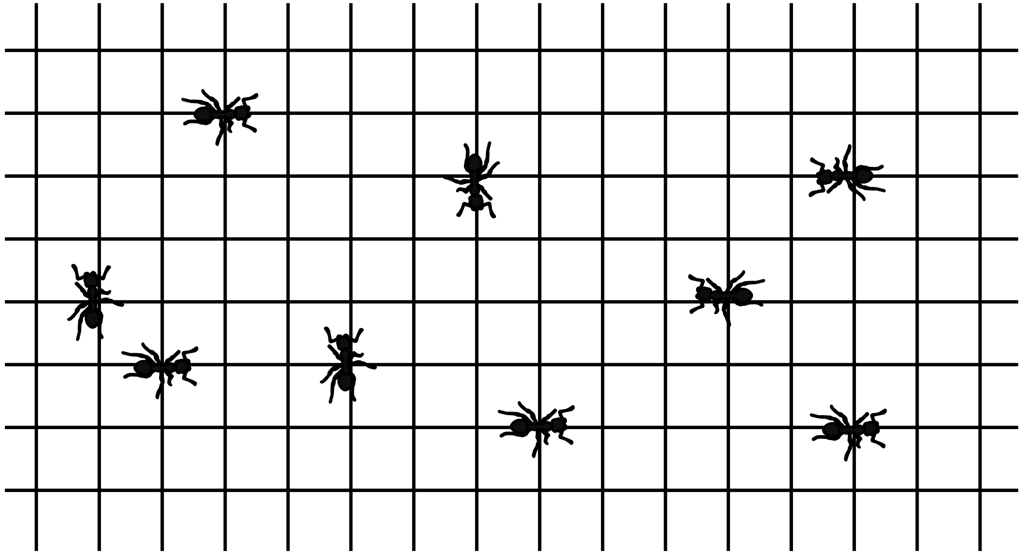

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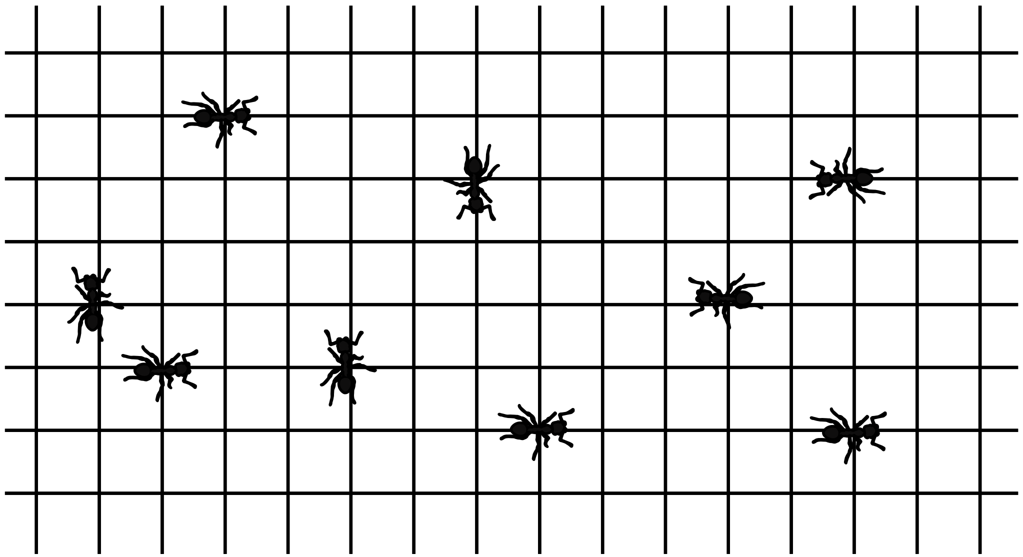

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Applications: size estimation for social networks, random-walk based sampling for sensor networks, density estimation for robot swarms.

Model Definition

- Underlying graph G (2-D torus).
- Each of the n ants is initially placed on a random node, independently from others.
- Discrete parallel time.
- At each round, each ant moves to a neighboring node chosen uniformaly at random (simple random walk).
- Only kind of interaction among ants is number of collisions $\tilde{d} = \sum_{r=1}^{t} c_j(r)$: an ant j count how many ants on her node at each time (no other info).
- The estimator is $\tilde{d} = \frac{1}{t} \sum_{r=1}^{t} c_j(r)$.

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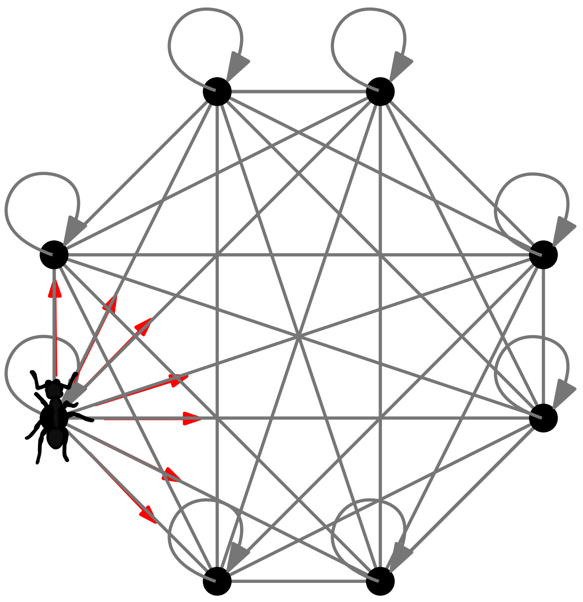
The mathematical challenge: after two ants meet, they are more likely to meet again.

$c_j(r')$ and $c_j(r'')$ are not independent!

Warm-up: Complete Graph

At each round each ants position is i.u.a.r.

$\Longrightarrow$ $c_j(r')$ and $c_j(r'')$ are independent!



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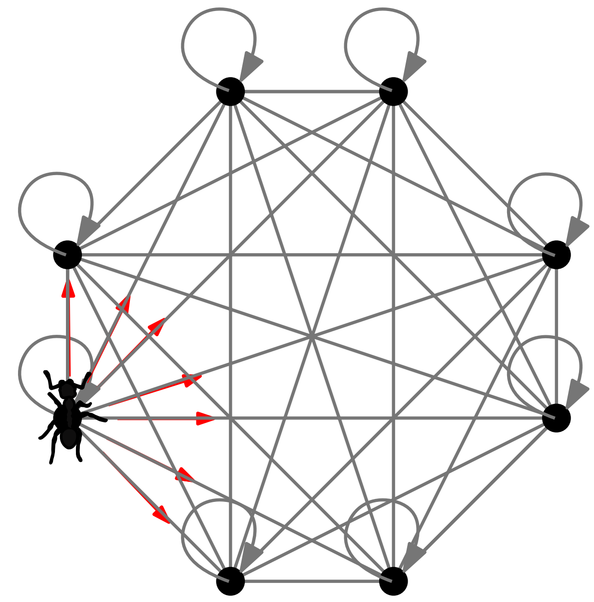

At each round each ants position is i.u.a.r.

$\implies$ $c_j(r')$ and $c_j(r'')$ are independent!

(A) Chernoff bound. Let $X_1, \ldots, X_N$ be independent 0-1 random variables with $\Pr(X_i = 1) = p$, then for any $\epsilon \in (0,1)$, $\Pr(|\sum_i X_i - Np| > \epsilon Np) \leq e^{-\frac{\epsilon^2}{3} Np}$.

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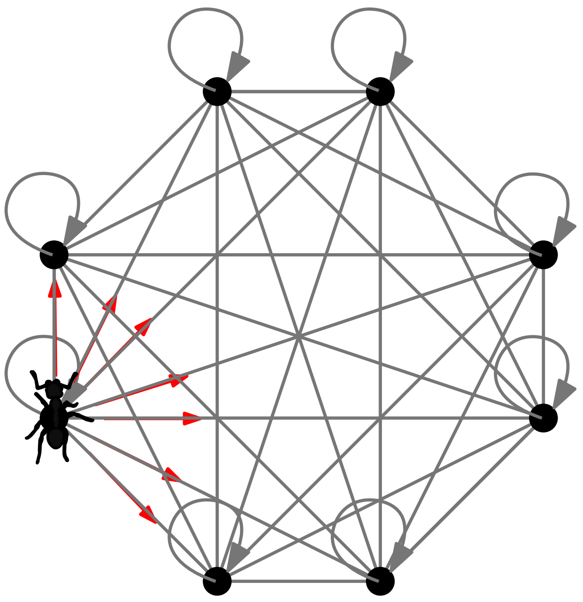

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$N = tn$, $X_{j,r} = c_j(r)$, $p = 1/A$, hence

$\Pr(|\tilde{d} - d| > \epsilon d) \leq e^{-\frac{\epsilon^2}{3}td} \leq \delta \implies \boxed{t = 3\log\frac{1}{\delta}/(d\epsilon^2)}$.

ABC of Concentration Inequalities

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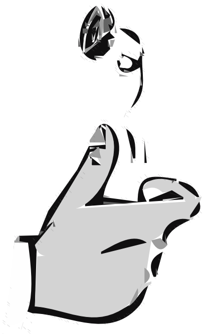

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For any non-decreasing function ψ,
$\Pr(X \geq t) = \Pr(\psi(X) \geq \psi(t)) \leq \mathbb{E}[\psi(X)]/\psi(t)$.

$X \leftarrow |X - \mathbb{E}X|$ and $\psi(x) = x^2 \implies$ Chebyshev inequality.

$X \leftarrow \sum_i X_i$ indip. and $\psi(X) = e^{-\lambda X} \implies$ Chernoff bounds.

A General Chernoff bound

General Chernoff bound (Chung-Lu). Let $X_1, ..., X_n$ be independent and $X_i \leq M$ for all i, then

$$\Pr\left(\sum_i X_i \geq \mathbb{E}\left(\sum_i X_i\right) + \Delta\right) \leq e^{-\frac{\Delta^2}{2\left(\sum_i \mathbb{E}\left(X_i^2\right) + M\Delta/3\right)}}.$$

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Proof.

$P(\sum_i X_i - \sum_i \mathbb{E}X_i > \Delta) \leq \mathbb{E}e^{\lambda \sum_i X_i}/e^{\Delta}$.

$Ee^{\lambda \sum_i X_i} = \prod_i Ee^{\lambda X_i}$.

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$Ee^{\lambda \sum_i X_i} = \prod_i Ee^{\lambda X_i}$.

Let $g(y) = 2\sum_{k=2}^{\infty} \frac{y^{k-2}}{k!} = \frac{2(e^y - 1 - y)}{y^2}$.

It holds $g(0) = 1$, $g(y) \leq 1$ for $y \geq 0$,
$g(y) = 2\sum_{k=2}^{\infty} \frac{y^{k-2}}{k!} \leq \sum_{k=2}^{\infty} \frac{y^{k-2}}{3^{k-2}} = \frac{1}{1-\frac{y}{3}}$ for $y < 3$ since $k! \geq 2 \cdot 3^{k-2}$.

A General Chernoff bound

We have

$$
\begin{aligned}
\mathbb{E}\left(e^{\lambda \sum_i X}\right) &= \prod_i \mathbb{E}\left(e^{\lambda X_i}\right) = \prod_i \mathbb{E}\left(\sum_{k=0}^{\infty} \frac{\lambda^k X_i^k}{k!}\right) \\
&= \prod_i \mathbb{E}\left(1 + \lambda \mathbb{E}\left(X_i\right) + \frac{1}{2}\lambda^2 X_i^2 g\left(\lambda X_i\right)\right) \\
&\leq \prod_i \left(1 + \lambda \mathbb{E}\left(X_i\right) + \frac{1}{2}\lambda^2 \mathbb{E}\left(X_i^2\right) g\left(\lambda M\right)\right) \\
&\leq \prod_i e^{\lambda \mathbb{E}(X_i) + \frac{1}{2}\lambda^2 \mathbb{E}\left(X_i^2\right) g(\lambda M)} \\
&= e^{\lambda \mathbb{E}\left(\sum_i X_i\right) + \frac{1}{2}\lambda^2 g(\lambda M) \sum_i \mathbb{E}\left(X_i^2\right)}.
\end{aligned}
$$

Hence, for λ satisfying $\lambda M < 3$, we have...

A General Chernoff bound

$$\Pr\left(\sum_i X_i \geq \mathbb{E}\left(\sum_i X_i\right) + \Delta\right) = \Pr\left(e^{\lambda X} \geq e^{\lambda \mathbb{E}\left(\sum_i X_i\right) + \lambda\Delta}\right)$$

$$\leq e^{-\lambda \mathbb{E}\left(\sum_i X_i\right) - \lambda\Delta} \mathbb{E}\left(e^{\lambda X}\right)$$

$$\leq e^{-\lambda\Delta + \frac{1}{2}\lambda^2 g(\lambda M) \sum_i \mathbb{E}\left(X_i^2\right)}$$

$$\leq e^{-\lambda\Delta + \frac{1}{2}\lambda^2 \frac{\sum_i \mathbb{E}\left(X_i^2\right)}{1-\lambda M/3}}.$$

Choosing $\lambda = \frac{\Delta}{\sum_i \mathbb{E}(X_i^2) + M\Delta/3}$, we have $\lambda M < 3$ and

$$\Pr\left(\sum_i X_i \geq \mathbb{E}\left(\sum_i X_i\right) + \Delta\right) \leq e^{-\lambda\Delta + \frac{1}{2}\lambda^2 \frac{\sum_i \mathbb{E}\left(X_i^2\right)}{1-\lambda M/3}}$$

$$\leq e^{-\frac{\Delta^2}{2\left(\sum_i \mathbb{E}\left(X_i^2\right) + M\Delta/3\right)}} \qquad \square$$

Main Result

Algorithm 1. Encounter Rate-Based Density Estimator

---

input: runtime t

$c := 0$

for $r = 1, ..., t$ **do**

$position = position + rand\{(0,1),(0,-1),(1,0),(-1,0)\}$

$c := c + count(position)$

end for

return $\tilde{d} = \frac{c}{t}$

Theorem. After running for t rounds, $t \leq A$, Algorithm 1 returns $\tilde{d}$ such that, for any $\delta > 0$, with prob $1-\delta$, $\delta d \in [(1-\epsilon)d, (1+\epsilon)d]$ for $\epsilon = \sqrt{\frac{\log\frac{1}{\delta}\log t}{td}}$. In other words, for any $\epsilon, \delta \in (0,1)$, if $t = \Theta(\frac{\log\frac{1}{\delta}\log\log\frac{1}{\delta}\log\frac{1}{d\epsilon}}{d\epsilon^2})$, $\tilde{d}$ is a $(1 \pm \epsilon)$ multiplicative estimate of d with probability $1-\delta$.

Proof Ingredients of Theorem 1

Re-collision Lemma. Consider two agents a_1 and a_2 randomly walking on a $\sqrt{A} \times \sqrt{A}$ torus. If a_1 and a_2 collide at time r, the prob. that they collide again in round $m + r$ is $\Theta(\frac{1}{m+1}) + \mathcal{O}(\frac{1}{A})$.

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Bernstein Inequality. Let $X_1, ..., X_n$ be independent random variables such that $\sum_i \mathbb{E}[X_i^2] \leq \nu$ and for each $k \geq 3$, $\sum_i \mathbb{E}[\max\{0, X_i\}^k] \leq \frac{k!}{2} \nu c^{k-2}$. Then $\Pr(\sum_i X_i - \sum_i \mathbb{E} X_i \geq t) \leq e^{-t^2/2(\nu + ct)}$.

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Remark. Proofs can be revisited to estimate probability that single random walk return on a given node (equalization).

Proof of Re-collision Lemma

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Two random walkers, a_1 and a_2.
Let M_x and M_y be the steps on x and y direction ($M_x + M_y = 2m$).
Let $\mathcal{C} =$"they collide again", and $\mathcal{C}_x$, and $\mathcal{C}_y$, the event that they end with same x, and y.

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$$\Pr(\mathcal{C} \mid M_x = m_x, M_y = m_y) = \Pr(\mathcal{C}_x \mid M_x = m_x) \Pr(\mathcal{C}_y \mid M_y = m_y).$$

Proof of Re-collision Lemma

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Wlog, we look at $\mathcal{C}_x$.
Let $\mathcal{C}_x^1$ and $\mathcal{C}_x^2$ be the events "same x without displacement" and "same x with displacement" (displacement=wrapping around torus), so
$\Pr(\mathcal{C}_x \mid M_x = m_x) = \Pr(\mathcal{C}_x^1 \mid M_x = m_x) + \Pr(\mathcal{C}_x^2 \mid M_x = m_x)$.

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The first summand means that the random walk comes back to the origin: $\Pr(\mathcal{C}_x^1 \mid M_x = m_x) = \binom{m_x}{m_x/2} (\frac{1}{2})^{m_x} = \frac{m_x!}{((m_x/2)!)^2} (\frac{1}{2})^{m_x}$.

Proof of Re-collision Lemma

Assuming m_x even and using Stirling $n! = \sqrt{2\pi n}(\frac{n}{e})^n(1 + \mathcal{O}(\frac{1}{n}))$, we get $\Pr(\mathcal{C}_x^1 \mid M_x = m_x) = \Theta(1/\sqrt{m_x + 1})$.

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As for C_x^2, $\Pr(\mathcal{C}_x^2 \mid M_x = m_x) = 2(\frac{1}{2})^{m_x} \sum_{c=1}^{\lfloor \frac{m_x}{\sqrt{A}} \rfloor} \binom{m_x}{(m_x - c\sqrt{A})/2}$.

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For $i \in [1, ..., \sqrt{A}-1]$,
let $\mathcal{D}_x^i =$"the walk is i steps clockwise from start after m_x steps". It holds

$$\Pr[\mathcal{D}_x^i | M_x = m_x] = \left(\frac{1}{2}\right)^{m_x} \cdot \sum_{c=-\left\lfloor \frac{m_x+i}{\sqrt{A}} \right\rfloor}^{\left\lfloor \frac{m_x-i}{\sqrt{A}} \right\rfloor} \binom{m_x}{\frac{m_x+i+c\sqrt{A}}{2}}$$

$$\geq \left(\frac{1}{2}\right)^{m_x} \cdot \sum_{c=-\left\lfloor \frac{m_x+i}{\sqrt{A}} \right\rfloor}^{-1} \binom{m_x}{\frac{m_x+i+c\sqrt{A}}{2}} = \left(\frac{1}{2}\right)^{m_x} \cdot \sum_{c=1}^{\left\lfloor \frac{m_x}{\sqrt{A}} \right\rfloor} \binom{m_x}{\frac{m_x+i-c\sqrt{A}}{2}}.$$

Proof of Re-collision Lemma

For any $i \in [1, ..., \sqrt{A} - 1]$, and any $c \geq 1$, $\frac{m_x+i-c\sqrt{A}}{2}$ is closer to $\frac{m_x}{2}$ than $\frac{m_x-c\sqrt{A}}{2}$ is, so

$$\binom{m_x}{\frac{m_x+i-c\sqrt{A}}{2}} > \binom{m_x}{\frac{m_x-c\sqrt{A}}{2}}$$

as long as $\frac{m_x+i-c\sqrt{A}}{2}$ is an integer. This allows us to lower bound $\Pr[\mathcal{D}_x^i | M_x = m_x]$ using $\Pr\left[\mathcal{C}_x^2 | M_x = m_x\right]$. Let $\mathcal{E}_{i,c}$ equal 1 if $\frac{m_x+i-c\sqrt{A}}{2}$ is an integer and 0 otherwise. Since $\mathcal{C}_x^2$ and each $\mathcal{D}_x^i$ are disjoint events:

$$\Pr\left[\mathcal{C}_x^2 | M_x = m_x\right] + \sum_{i=1}^{\sqrt{A}-1} \Pr\left[\mathcal{D}_x^i | M_x = m_x\right] \leq 1$$

$$\Pr\left[\mathcal{C}_x^2 | M_x = m_x\right] + \left(\frac{1}{2}\right)^{m_x} \cdot \sum_{i=1}^{\sqrt{A}-1} \left(\sum_{c=1}^{\left\lfloor \frac{m_x}{\sqrt{A}} \right\rfloor} \binom{m_x}{\frac{m_x+i-c\sqrt{A}}{2}} \right) \leq 1$$

Proof of Re-collision Lemma

$$\Pr\left[\mathcal{C}_x^2 | M_x = m_x\right] + \left(\frac{1}{2}\right)^{m_x} \cdot \sum_{c=1}^{\left\lfloor \frac{m_x}{\sqrt{A}} \right\rfloor} \left(\binom{m_x}{\frac{m_x - c\sqrt{A}}{2}} \cdot \sum_{i=1}^{\sqrt{A}-1} \mathcal{E}_{i,c} \right) \leq 1$$

$$\Pr\left[\mathcal{C}_x^2 | M_x = m_x\right] \cdot \Theta(\sqrt{A}) \leq 1.$$

The last step follows from combining the last with the fact that $\sum_{i=1}^{\sqrt{A}-1} \mathcal{E}_{i,c} = \Theta\left(\sqrt{A}\right)$ for all c since $\frac{m_x + i - c\sqrt{A}}{2}$ is integral for half the possible $i \in [1, ..., \sqrt{A} - 1]$. Rearranging, we have $\Pr\left[\mathcal{C}_x^2 | M_x = m_x\right] = O\left(\frac{1}{\sqrt{A}}\right)$.

Proof of Re-collision Lemma

Combining our bounds for $\mathcal{C}_x^1$ and $\mathcal{C}_x^2$,
$\Pr\left[\mathcal{C}_x | M_x = m_x\right] = \Theta\left(\frac{1}{\sqrt{m_x+1}}\right) + O\left(\frac{1}{\sqrt{A}}\right).$

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Identical bounds hold for the y direction and by saparating horizantal/vertical axis we have:

$$\Pr\left[\mathcal{C} | M_x = m_x, M_y = m_y\right] = \Theta\left(\frac{1}{\sqrt{(m_x+1)(m_y+1)}}\right) + O\left(\frac{1}{\sqrt{A(m_x+1)}} + \frac{1}{\sqrt{A(m_y+1)}}\right) + O\left(\frac{1}{A}\right).$$

Proof of Re-collision Lemma

Our final step is to remove the conditioning on M_x and M_y. Since direction is chosen independently and uniformly at random for each step, $\mathbf{E}\left[M\right]_x = \mathbf{E}\left[M\right]_y = m$. By a standard Chernoff bound:

$$\Pr[M_x \leq m/2] \leq 2e^{-(1/2)^2 \cdot m/2} = O\left(\frac{1}{m+1}\right).$$

(Again using $m+1$ instead of m to cover the $m=0$ case). An identical bound holds for M_y, and so, except with probability $O\left(\frac{1}{m+1}\right)$ both are $\geq m/2$. We get:

$$\begin{aligned}\Pr\left[\mathcal{C}\right] &= \Theta\left(\frac{1}{m+1}\right) + O\left(\frac{1}{\sqrt{A(m+1)}}\right) + O\left(\frac{1}{A}\right) \\ &= \Theta\left(\frac{1}{m+1}\right) + O\left(\frac{1}{A}\right). \qquad \square\end{aligned}$$

Proof of First Collision Lemma

First-collision Lemma. Assuming $t \leq A$, for all $j \in [1, ..., n]$, $\Pr[c_j \geq 1] = \Theta(\frac{t}{A \log t})$.

Using the fact that c_j is identically distributed for all j,

$$\mathbb{E}\tilde{d} = d = \frac{1}{t} \cdot \mathbb{E} \sum_{i=1}^{n} c_i = \frac{n}{t} \cdot \mathbf{E}\left[c\right]_j = \frac{n}{t} \cdot \Pr\left[c_j \geq 1\right] \cdot \mathbb{E}[c_j | c_j \geq 1]$$

$$\frac{n}{A} = \frac{n}{t} \cdot \Pr\left[c_j \geq 1\right] \cdot \mathbb{E}[c_j | c_j \geq 1].$$

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$$\frac{n}{A} = \frac{n}{t} \cdot \Pr\left[c_j \ge 1\right] \cdot \mathbb{E}[c_j | c_j \ge 1].$$

Rearranging gives:

$$\Pr\left[c_j \ge 1\right] = \frac{t}{A \cdot \mathbb{E}[c_j | c_j \ge 1]}.$$

Proof of First Collision Lemma

To compute $\mathbb{E}[c_j|c_j \geq 1]$, we use Re-collision Lemma and linearity of expectation. Since $t \leq A$, the $O\left(\frac{1}{A}\right)$ term in Re-collision Lemma is absorbed into the $\Theta\left(\frac{1}{m+1}\right)$. Let $r \leq t$ be the first round that the two agents collide. We have:

$$\mathbb{E}[c_j|c_j \geq 1] = \sum_{m=0}^{t-r} \Theta\left(\frac{1}{m+1}\right) = \Theta\left(\log(t-r)\right).$$

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$$\mathbb{E}[c_j|c_j \geq 1] = \sum_{m=0}^{t-r} \Theta\left(\frac{1}{m+1}\right) = \Theta\left(\log(t-r)\right).$$

After any round the agents are located at uniformly and independently chosen positions, so collide with probability exactly $1/A$. So, the probability of the *first* collision between the agents being in a given round can only decrease as the round number increases. So, at least $1/2$ of the time that $c_j \geq 1$, there is a collision in the first $t/2$ rounds. So, overall, thanks to the previous calculations, $\mathbb{E}[c_j|c_j \geq 1] = \Theta\left(\log(t - t/2)\right) = \Theta\left(\log t\right)$, hence $\Pr\left[c_j \geq 1\right] = \Theta\left(\frac{t}{A \cdot \log t}\right)$, completing the proof. $\square$

Proof of Collision Moment Lemma

Collision Moment Lemma. For $j \in [1, ..., n]$, let $\bar{c}_j \stackrel{def}{=} c_j - {}_j$. For all $k \geq 2$, assuming $t \leq A$, $\mathbb{E}[\bar{c}_j^k] = \mathcal{O}(\frac{t}{A} k! \log^{k-1} t)$.

We expand $\mathbb{E}[\bar{c}_j^k] = \Pr[c_j \geq 1] \cdot \mathbb{E}[\bar{c}_j^k | c_j \geq 1] + \Pr[c_j = 0] \cdot \mathbb{E}[\bar{c}_j^k | c_j = 0]$, and so by First Collision Lemma:

$$\mathbb{E}\left[\bar{c}_j^k\right] = O\left(\frac{t}{A \log t} \cdot \mathbb{E}\left[\bar{c}_j^k | c_j \geq 1\right] + \mathbb{E}\left[\bar{c}_j^k | c_j = 0\right]\right).$$

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$\mathbb{E}\left[\bar{c}_j^k | c_j = 0\right] = (\mathbb{E} c_j)^k = (t/A)^k \leq \frac{t}{A} k! \log^{k-1} t$ for all $k \geq 2$. Further, $\mathbb{E}\left[\bar{c}_j^k | c_j \geq 1\right] \leq \mathbb{E}\left[c_j^k | c_j \geq 1\right]$, since $\mathbb{E} c_j = \frac{t}{A} \leq 1$. So to prove the lemma, it just remains to show that $\mathbb{E}\left[c_j^k | c_j \geq 1\right] = O\left(k! \log^k t\right)$.

Proof of Collision Moment Lemma

Conditioning on $c_j \geq 1$, we know the agents have an initial collision in some round $t' \leq t$. We split c_j over rounds:
$c_j = \sum_{r=t'}^{t} c_j(r) \leq \sum_{r=t'}^{t'+t-1} c_j(r)$. To simplify notation we relabel round t' round 1 and so round $t'+t-1$ becomes round t. Expanding c_j^k out fully using the summation:

$$\begin{aligned}
\mathbb{E}\left[c_j^k\right] &= \mathbb{E}\left[\sum_{r_1=1}^{t}\sum_{r_2=1}^{t}\dots\sum_{r_k=1}^{t} c_j(r_1)c_j(r_2)\dots c_j(r_k)\right] \\
&= \sum_{r_1=1}^{t}\sum_{r_2=1}^{t}\dots\sum_{r_k=1}^{t} \mathbb{E}\left[c_j(r_1)c_j(r_2)\dots c_j(r_k)\right].
\end{aligned}$$

Proof of Collision Moment Lemma

$\mathbb{E}\left[c_{r_1}(j)c_{r_2}(j)...c_{r_k}(j)\right]$ is just the probability that the two agents collide in each of rounds $r_1, r_2, ...r_k$. Assume w.l.o.g. that $r_1 \leq r_2 \leq ... \leq r_k$. By Re-collision Lemma this is: $O\left(\frac{1}{r_1(r_2-r_1+1)(r_3-r_2+1)...(r_k-r_{k-1}+1)}\right)$. So we can rewrite, by linearity of expectation:

$$] = k! \sum_{r_1=1}^{t} \sum_{r_2=r_1}^{t} ... \sum_{r_k=r_{k-1}}^{t} O\left(\frac{1}{r_1(r_2-r_1+1)(r_3-r_2+1)...(r_k-r_{k-1}+1)}\right).$$

Proof of Collision Moment Lemma

The $k!$ comes from the fact that in this sum we only have *ordered* k-tuples and so need to multiple by $k!$ to account for the fact that the original sum is over *unordered* k-tuples. We can bound:

$$\sum_{r_k=r_{k-1}}^{t} \frac{1}{r_k - r_{k-1} + 1} = 1 + \frac{1}{2} + ... + \frac{1}{t} = O(\log t)$$

so rearranging the sum and simplifying gives:

$$\begin{aligned}
\mathbb{E}\left[c_j^k\right] &= k! \sum_{r_1=1}^{t} \frac{1}{r_1} \sum_{r_2=r_1+1}^{t} \frac{1}{r_2 - r_1} ... \sum_{r_k=r_{k-1}+1}^{t} \frac{1}{r_k - r_{k-1}} \\
&= k! \sum_{r_1=1}^{t} \frac{1}{r_1} \sum_{r_2=r_1}^{t} \frac{1}{r_2 - r_1 + 1} ... \sum_{r_{k-1}=r_{k-2}}^{t} \frac{1}{r_{k-2} - r_{k-1} + 1} \cdot O(\log t).
\end{aligned}$$

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so rearranging the sum and simplifying gives:

$$\begin{aligned}
\mathbb{E}\left[c_j^k\right] &= k! \sum_{r_1=1}^{t} \frac{1}{r_1} \sum_{r_2=r_1+1}^{t} \frac{1}{r_2 - r_1} \ldots \sum_{r_k=r_{k-1}+1}^{t} \frac{1}{r_k - r_{k-1}} \\
&= k! \sum_{r_1=1}^{t} \frac{1}{r_1} \sum_{r_2=r_1}^{t} \frac{1}{r_2 - r_1 + 1} \ldots \sum_{r_{k-1}=r_{k-2}}^{t} \frac{1}{r_{k-2} - r_{k-1} + 1} \cdot O(\log t).
\end{aligned}$$

We repeat this simplification for each level of summation replacing $\sum_{r_i=r_{i-1}}^{t} \frac{1}{r_i - r_{i-1}+1}$ with $O(\log t)$. Iterating through the k levels gives $\mathbb{E}\left[c_j^k\right] = O(k! \log^k t)$ giving the lemma. $\square$


MERRY
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