On the Computational Power of Simple Dynamics

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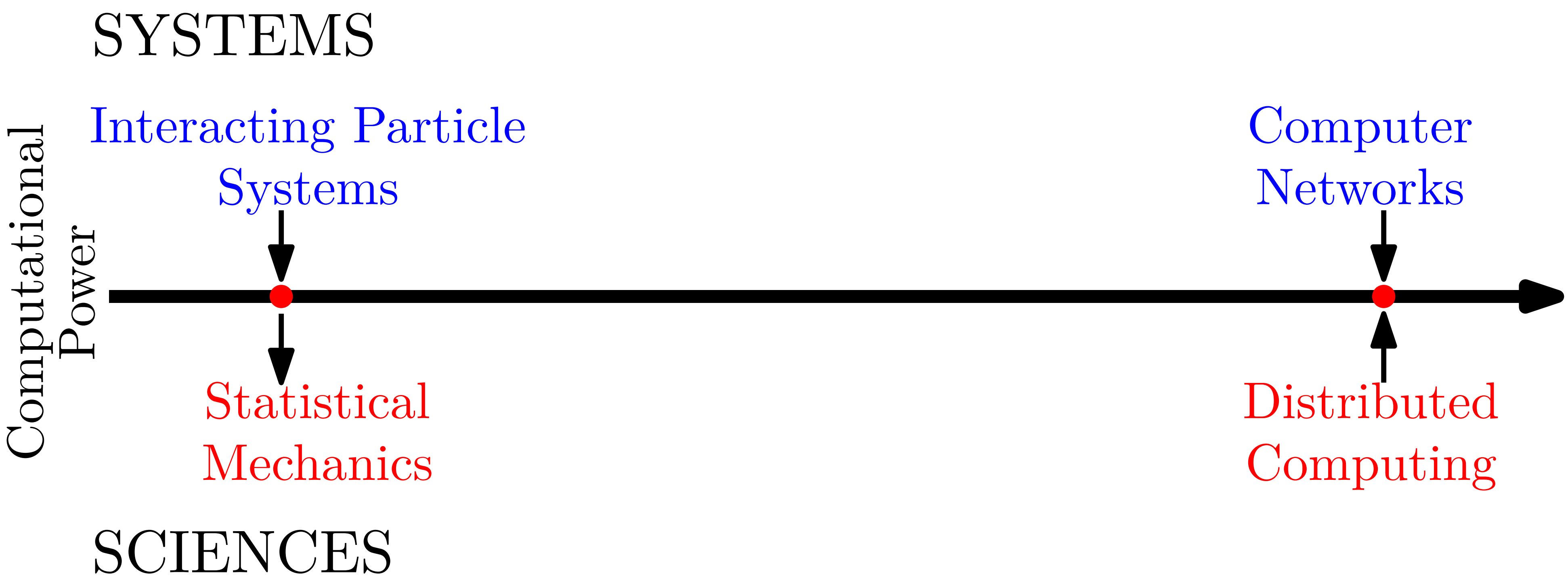
Sapienza University of Rome
February 13, 2017

Communication in *Simple* Systems



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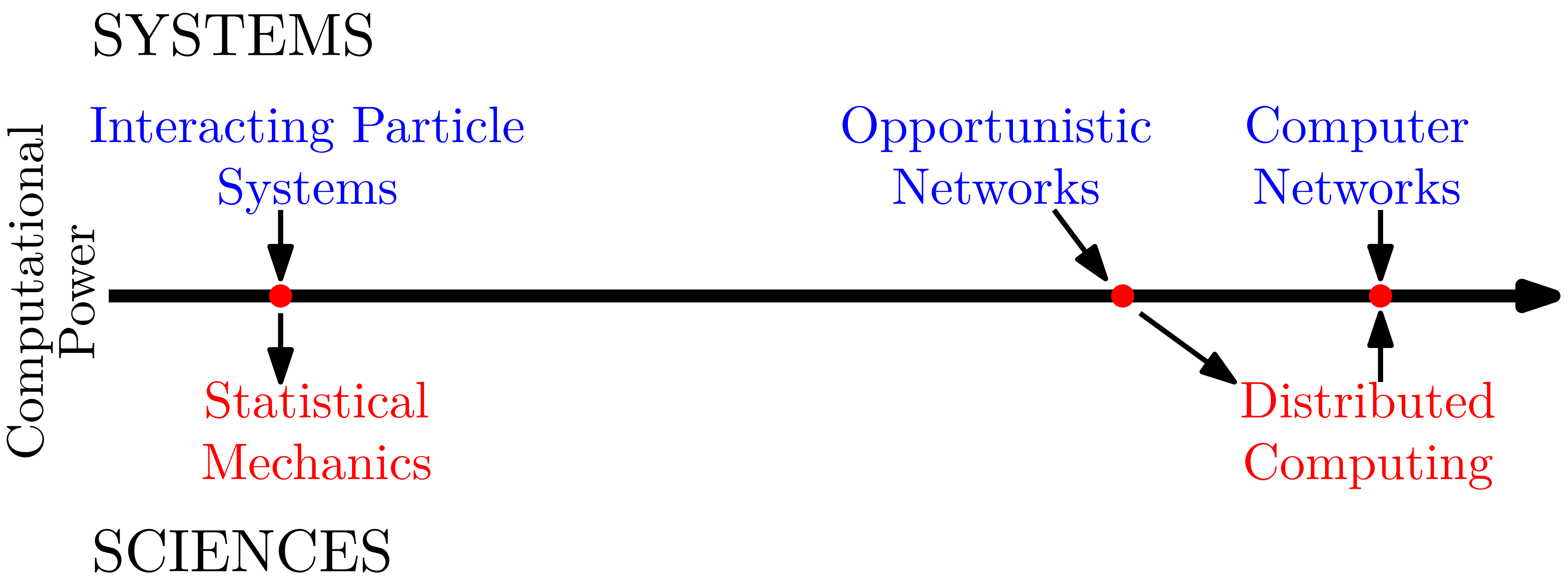

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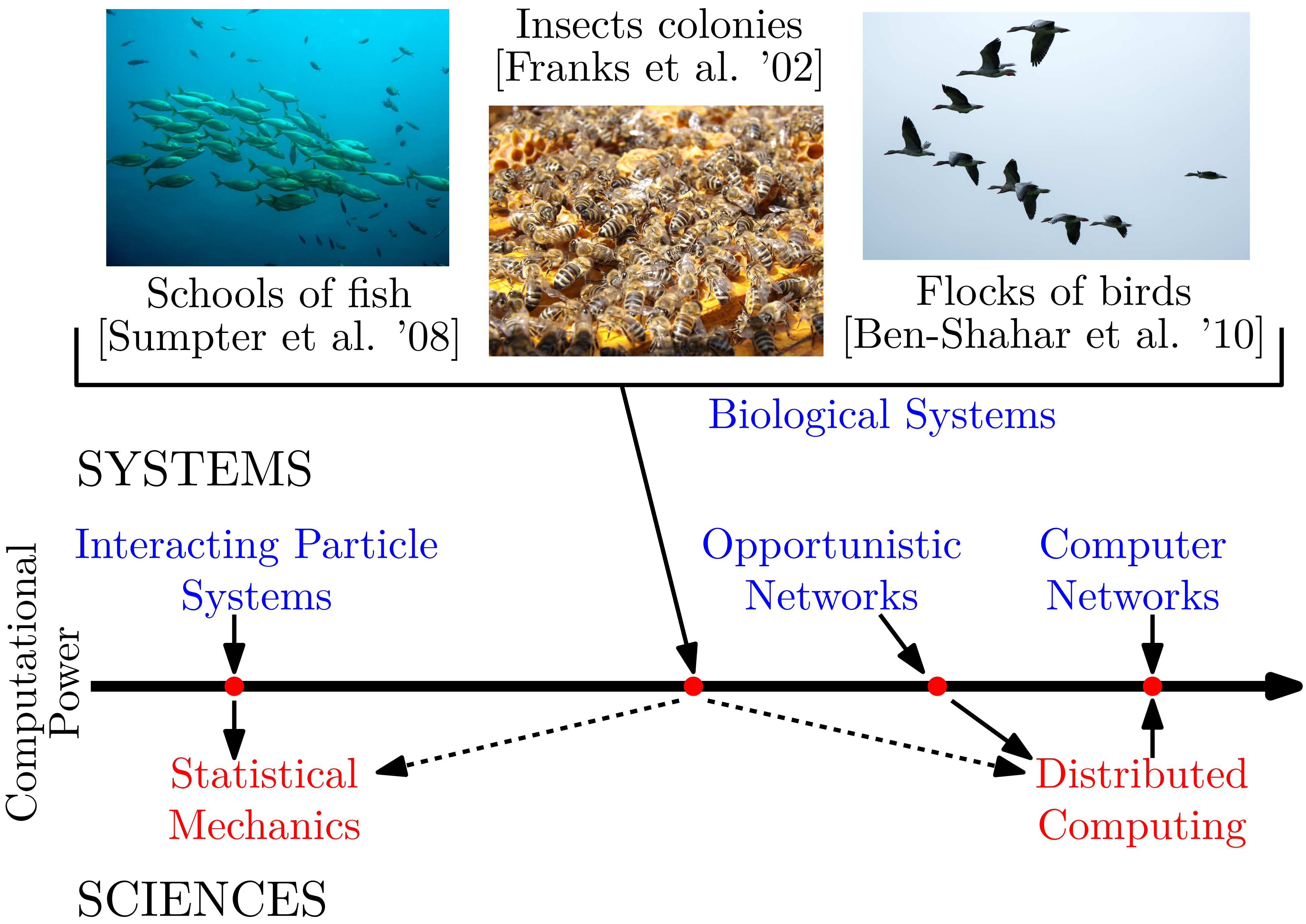

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Animal communication:

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- Anonymous
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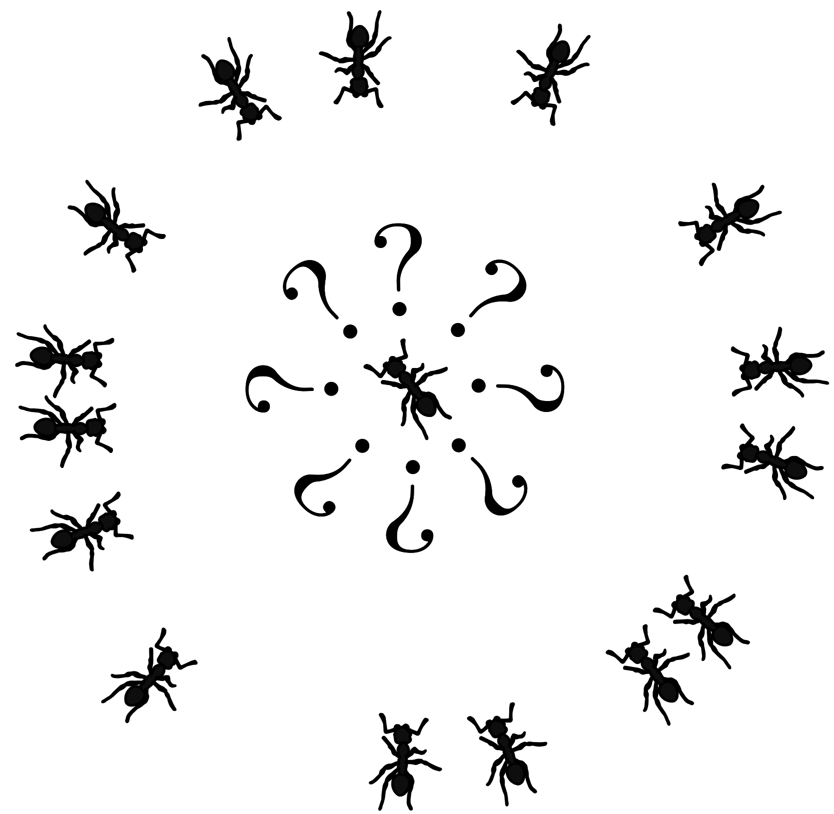

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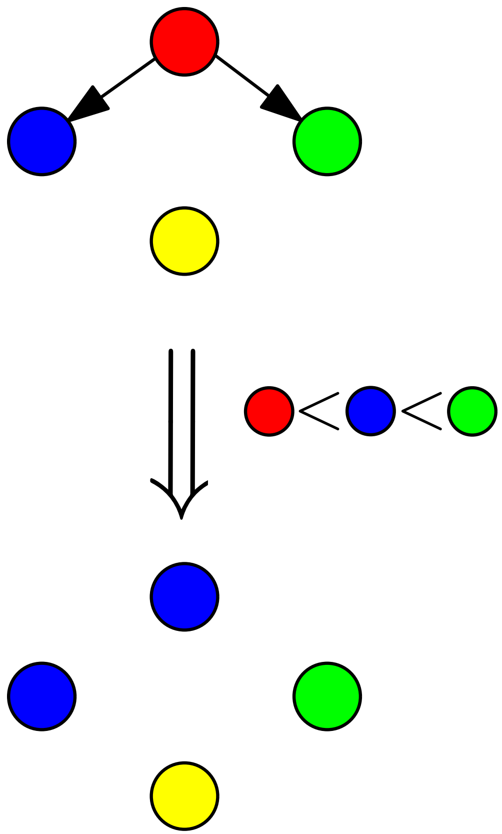

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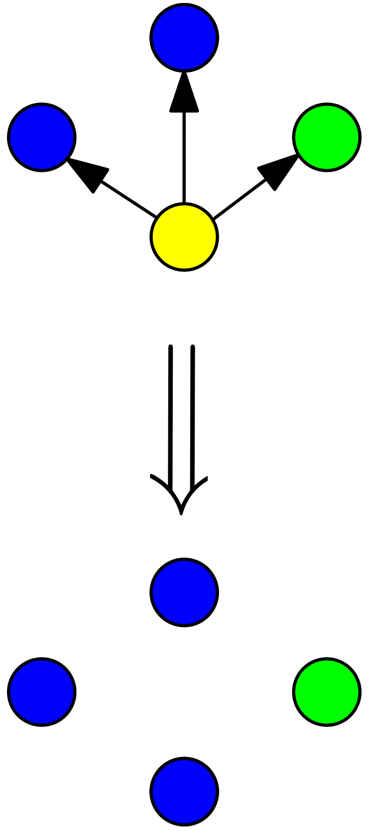

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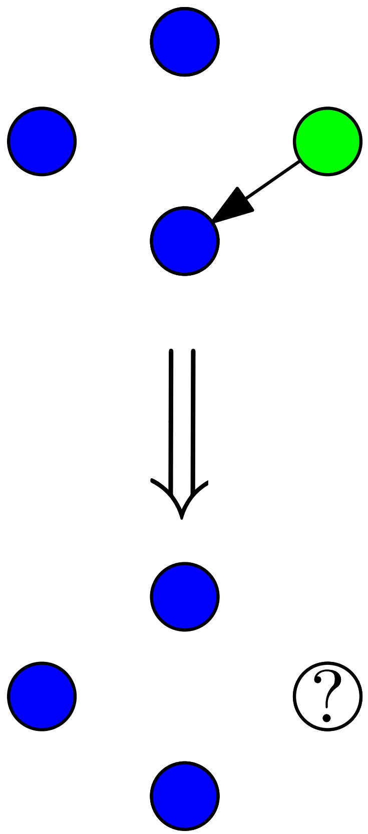

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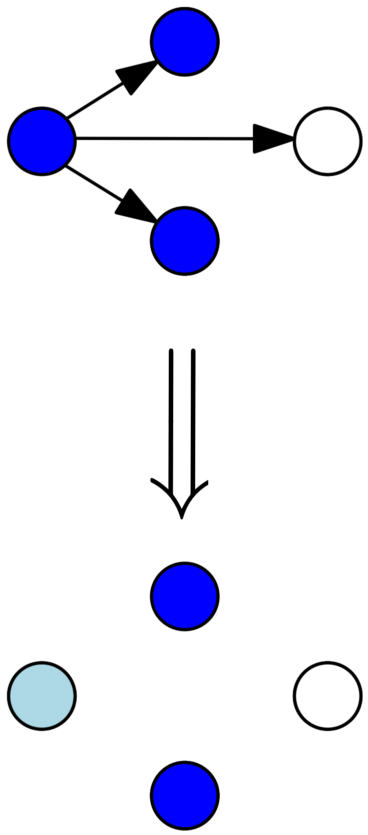

Examples of Dynamics

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- Averaging dynamics

The Power of Dynamics: Plurality Consensus

Computing the Median

3-Median dynamics [Doerr et al. '11]. Converge to $\mathcal{O}(\sqrt{n \log n})$ approximation of median of system in $\mathcal{O}(\log n)$ rounds w.h.p., even if $\mathcal{O}(\sqrt{n})$ states are arbitrarily changed at each round ($\mathcal{O}(\sqrt{n})$-bounded adversary).

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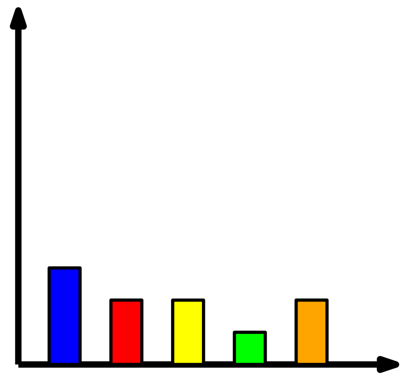
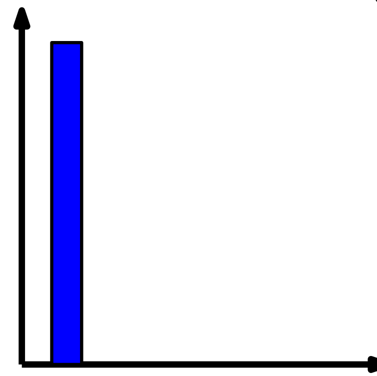
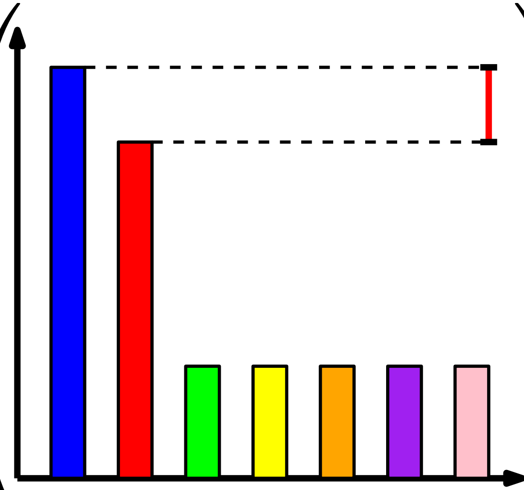
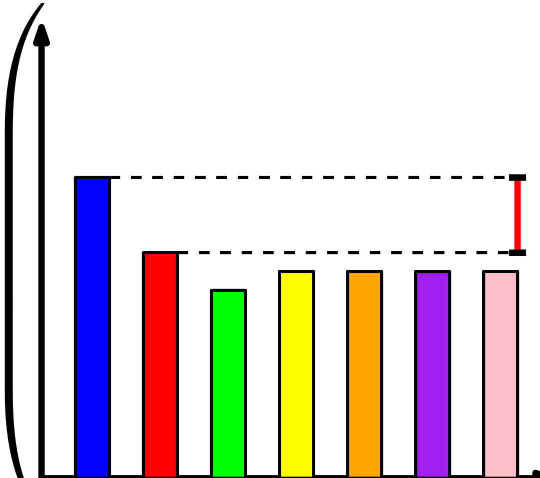
Undecided-State dynamics [SODA '15]. If majority/second-majority ($c_{maj}/c_{2^{nd}maj}$) is at least $1 + \epsilon$, system converges to plurality within $\tilde{\Theta}(\text{md}(\mathbf{c}))$ rounds w.h.p.

A Global Measure of Bias

$$\mathrm{md}(\mathbf{c}^{(\mathbf{0})}) := \sum_{i=1}^{k} \left(\frac{c_i^{(0)}}{c_{maj}^{(0)}} \right)^2 = 1 + \mathcal{D}\Bigg($$  ,  $$\Bigg)$$

$$1 \leq \mathrm{md}\Bigg($$  $$\Bigg) \ll \mathrm{md}\Bigg($$  $$\Bigg) \leq k$$

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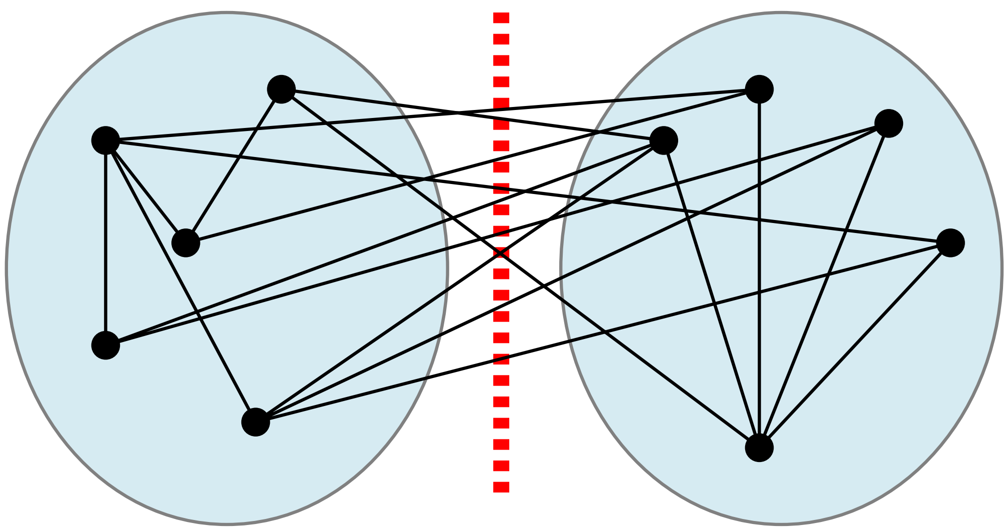
Can dynamics solve a problem non-trivial in centralized setting?

Community Detection as Minimum Bisection

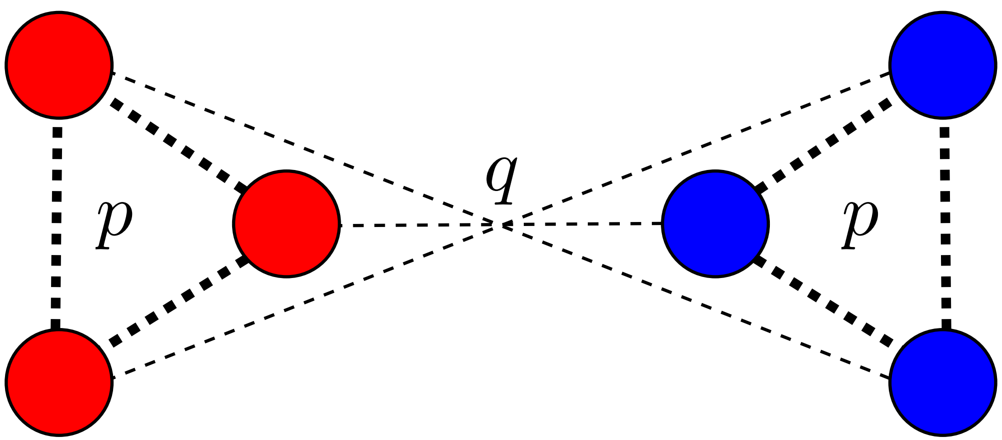
Minimum Bisection Problem.
Input: a graph G with $2n$ nodes.
Output: $S = \arg\min_{\substack{S \subset V \\ |S| = n}} E(S, V - S)$.



[Garey, Johnson, Stockmeyer '76]:
Min-Bisection is *NP-Complete*.

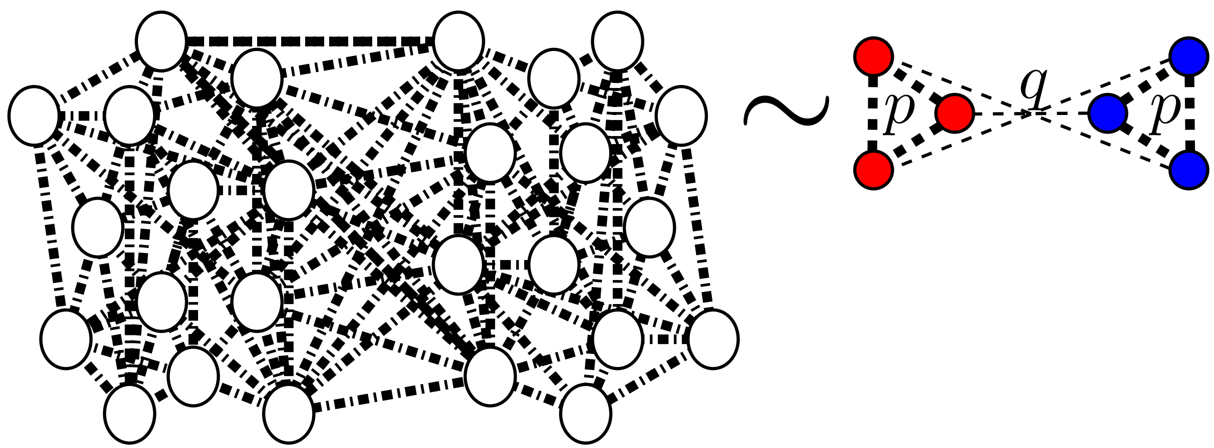
The Stochastic Block Model

Stochastic Block Model (SBM). Two "communities" of equal size V_1 and V_2, each edge inside a community included with probability $p = \frac{a}{n}$, each edge across communities included with probability $q = \frac{b}{n} < p$.



The Stochastic Block Model

Reconstruction problem. Given graph generated by SBM, find original partition.



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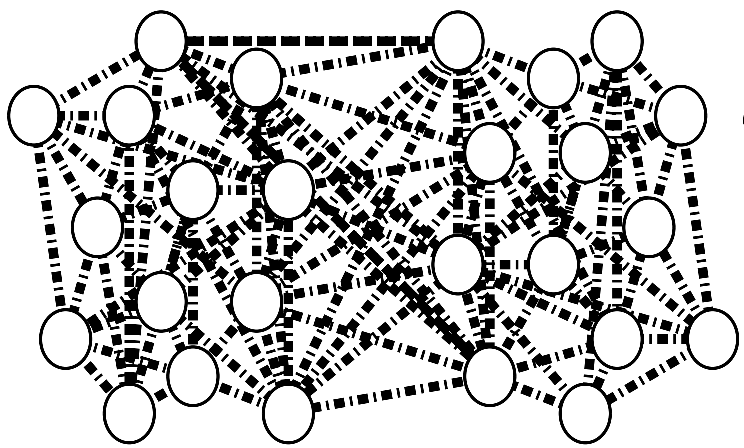

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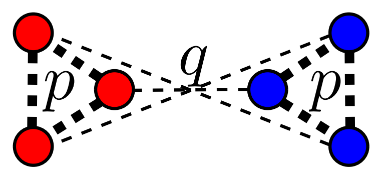
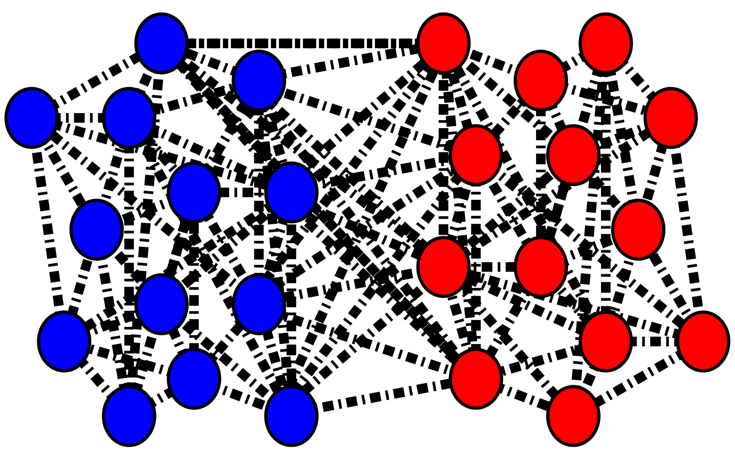
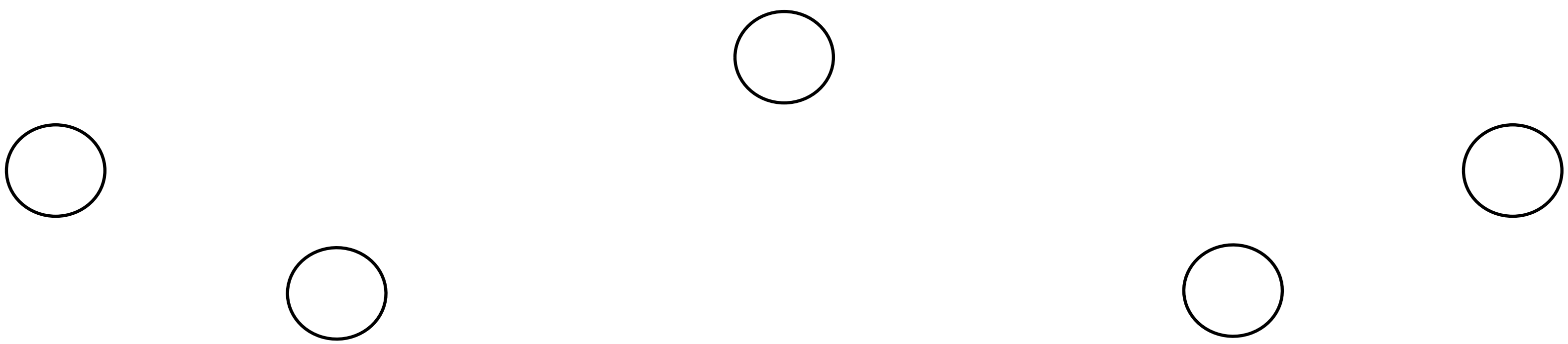




The Averaging Dynamics

Al nodes at the same time:

- At $t = 0$, randomly pick value $x^{(t)} \in \{\mathbf{+1}, \mathbf{-1}\}$.
- Then, at each round
  1. Set value $x^{(t)}$ to average of neighbors,
  2. Set label to **blue** if $x^{(t)} < x^{(t-1)}$, **red** otherwise.



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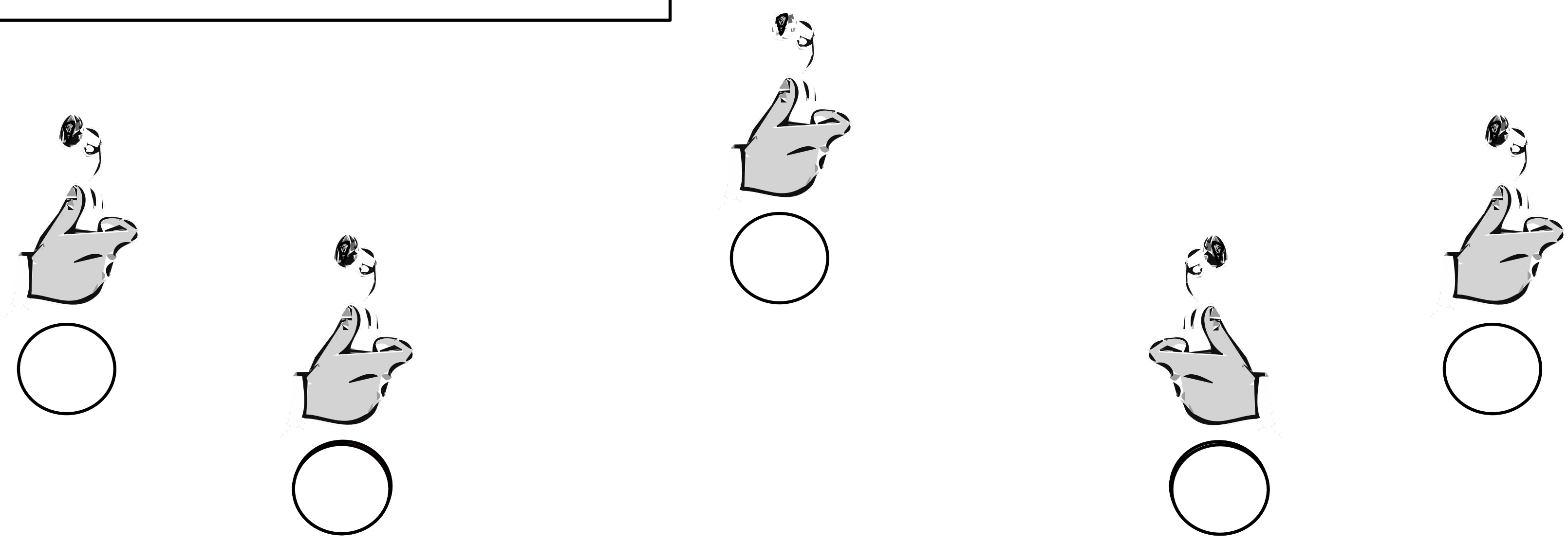

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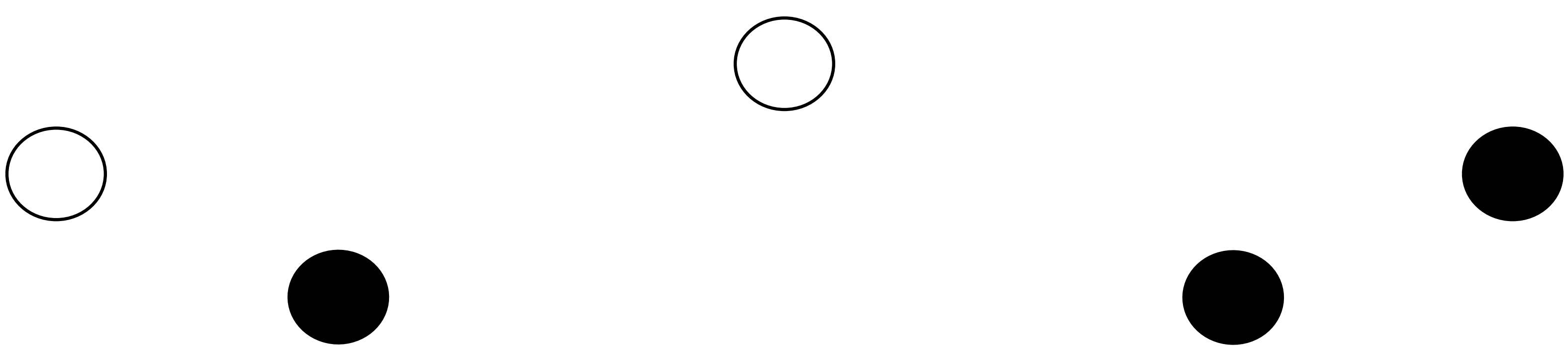

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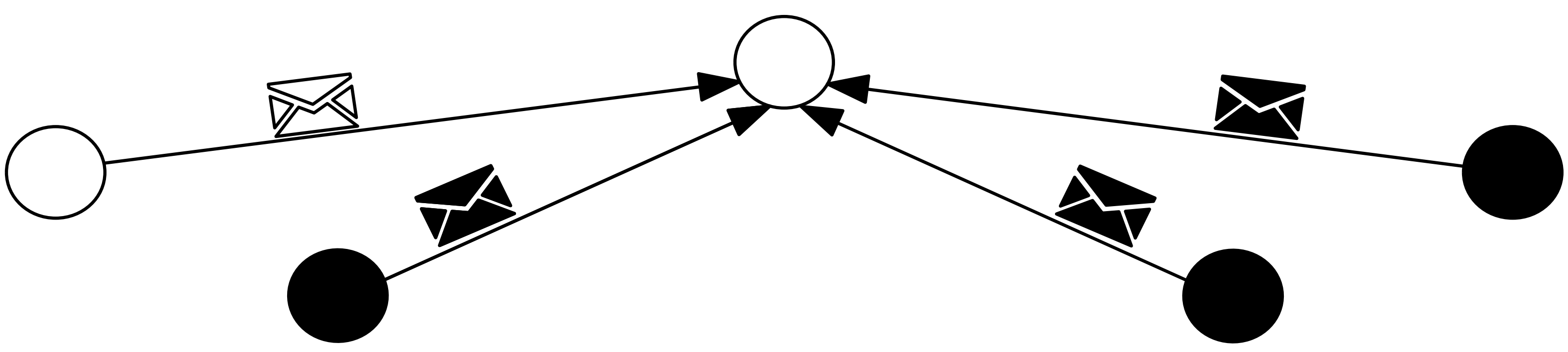

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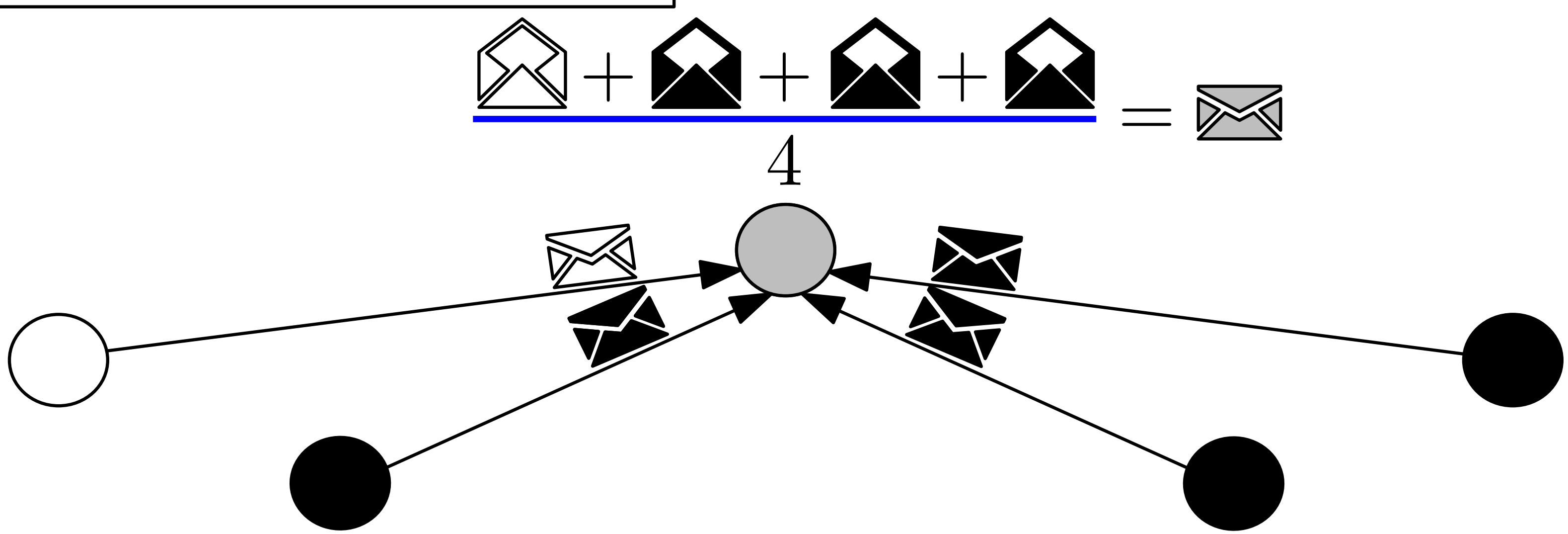

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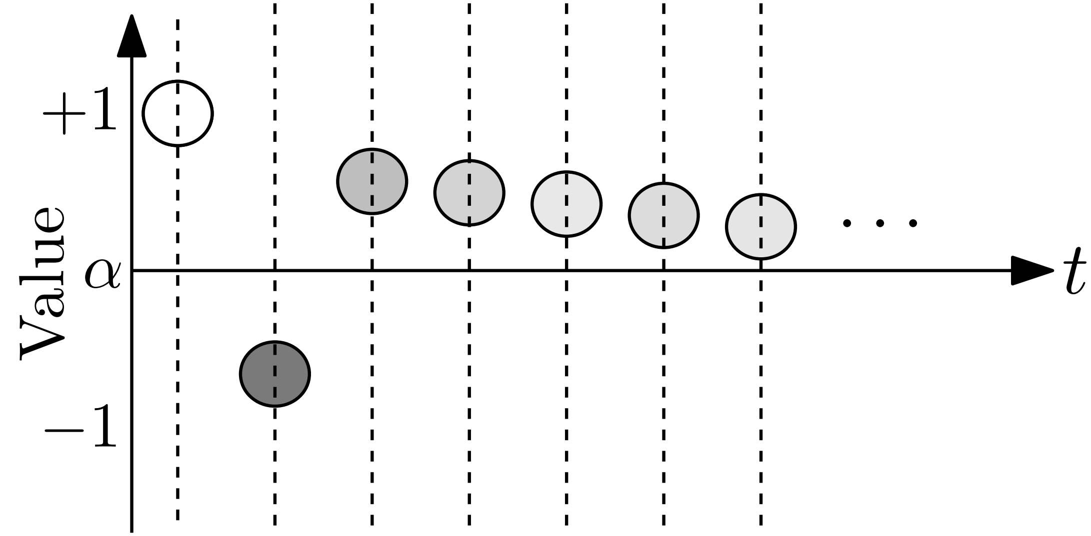

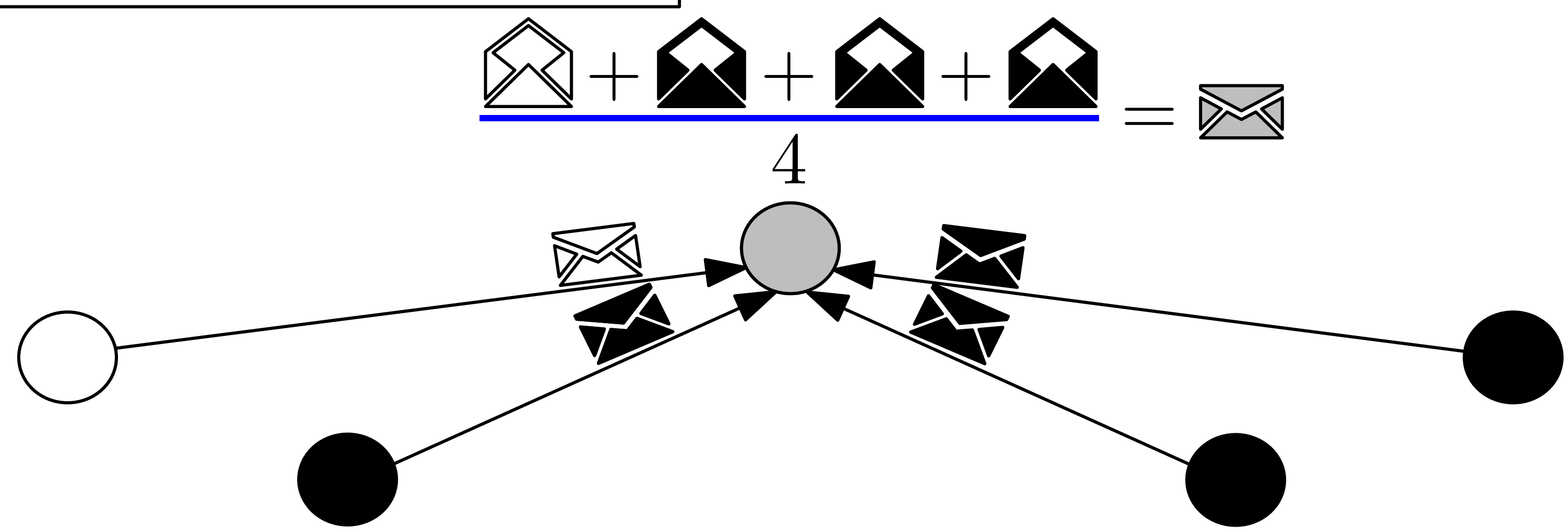

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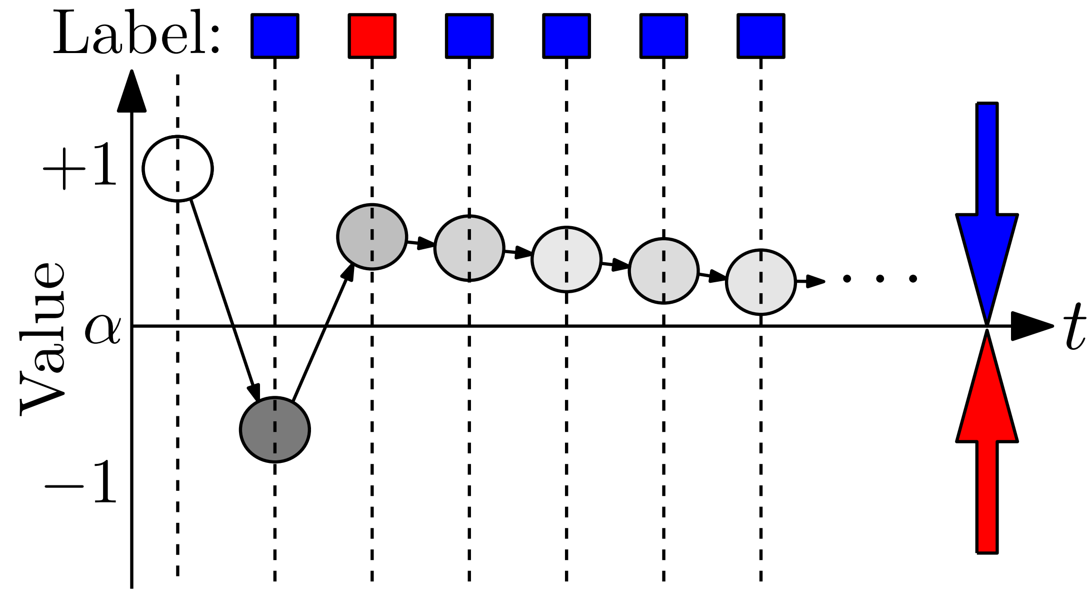

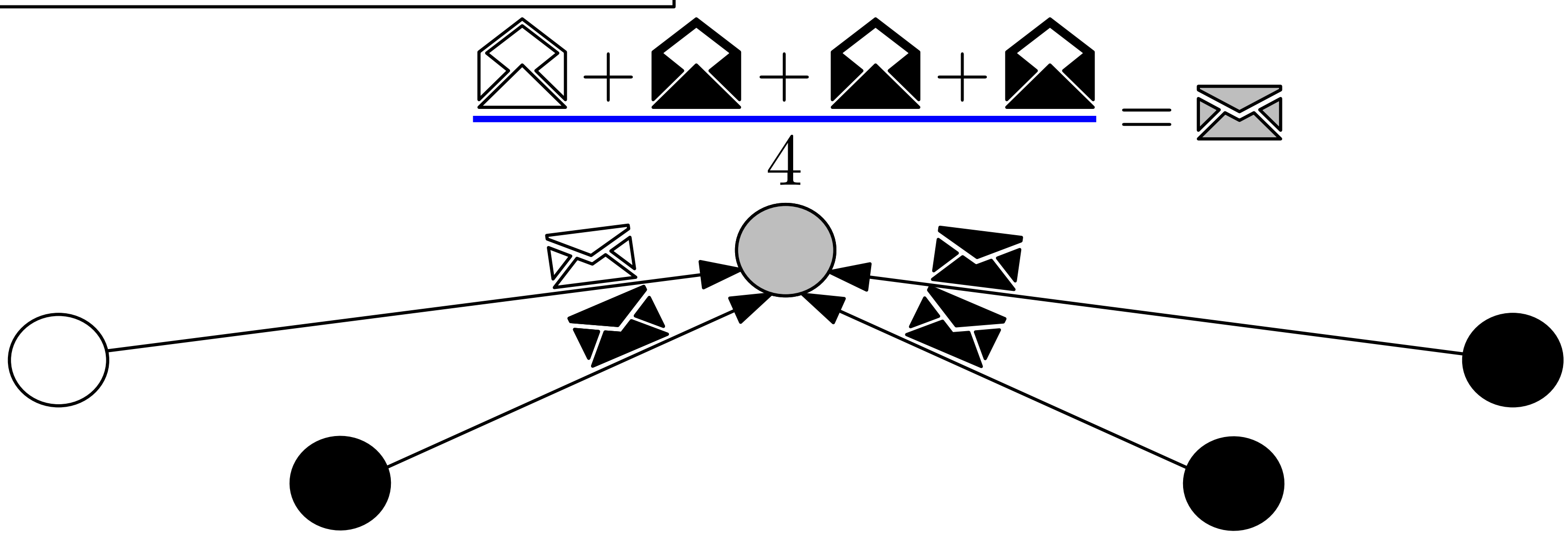

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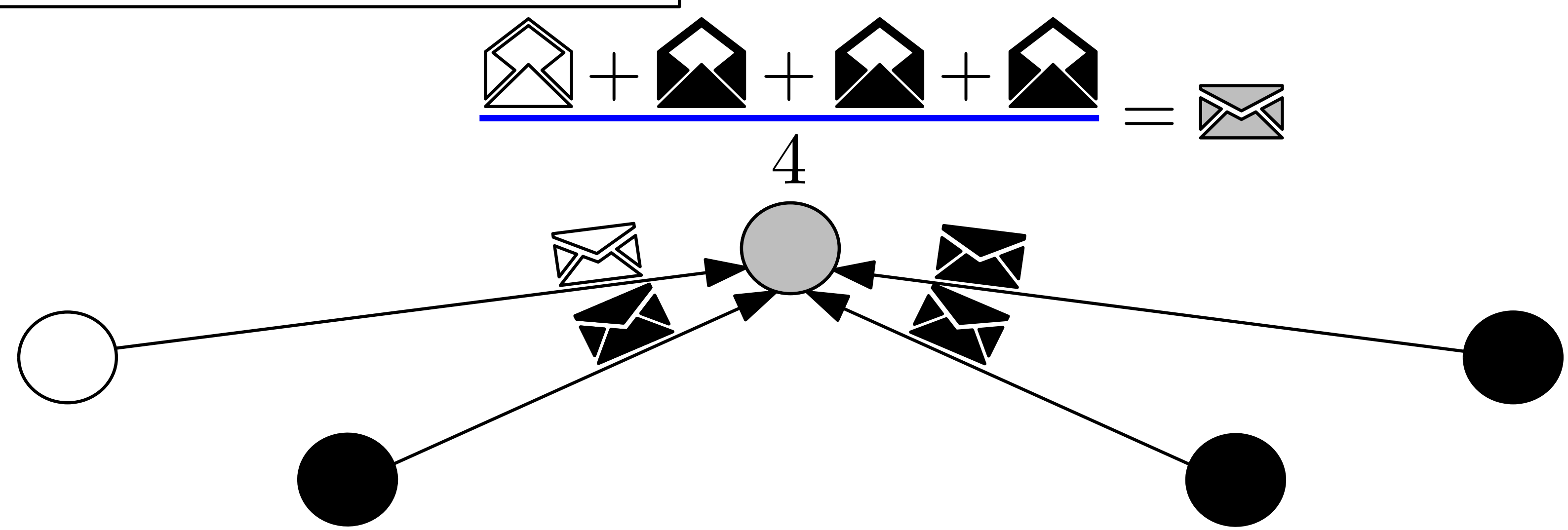
- Converges to (weighted) global average of initial values,
- Convergence time = mixing time of G,
- Important applications in fault-tolerant self-stabilizing consensus.



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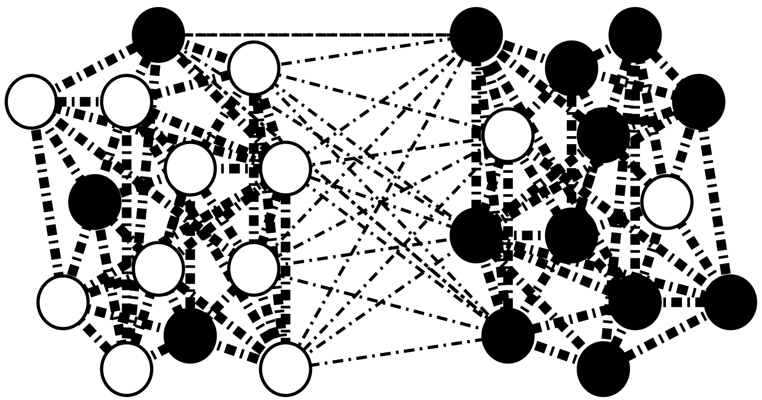
Averaging is a **linear** dynamics

$$\mathbf{x}^{(t)} = \begin{pmatrix} \circ \\ \bullet \\ \circ \\ \bullet \\ \bullet \end{pmatrix}$$

$$\boxed{\mathbf{x}^{(t)} = P \cdot \mathbf{x}^{(t-1)} = P^t \cdot \mathbf{x}^{(0)}}$$

P transition matrix of random walk

Community Detection via Averaging Dynamics



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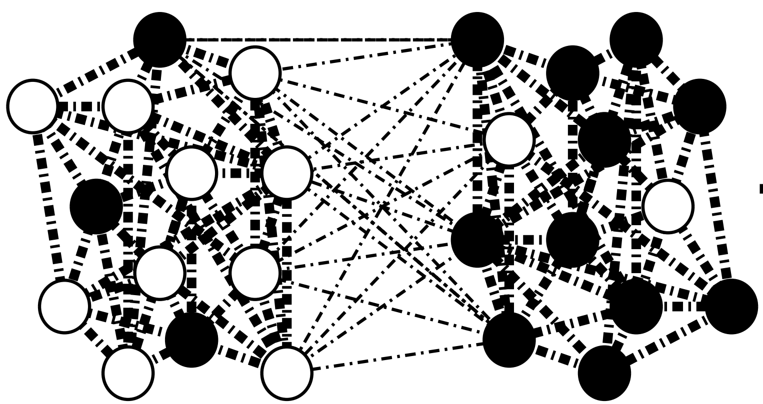

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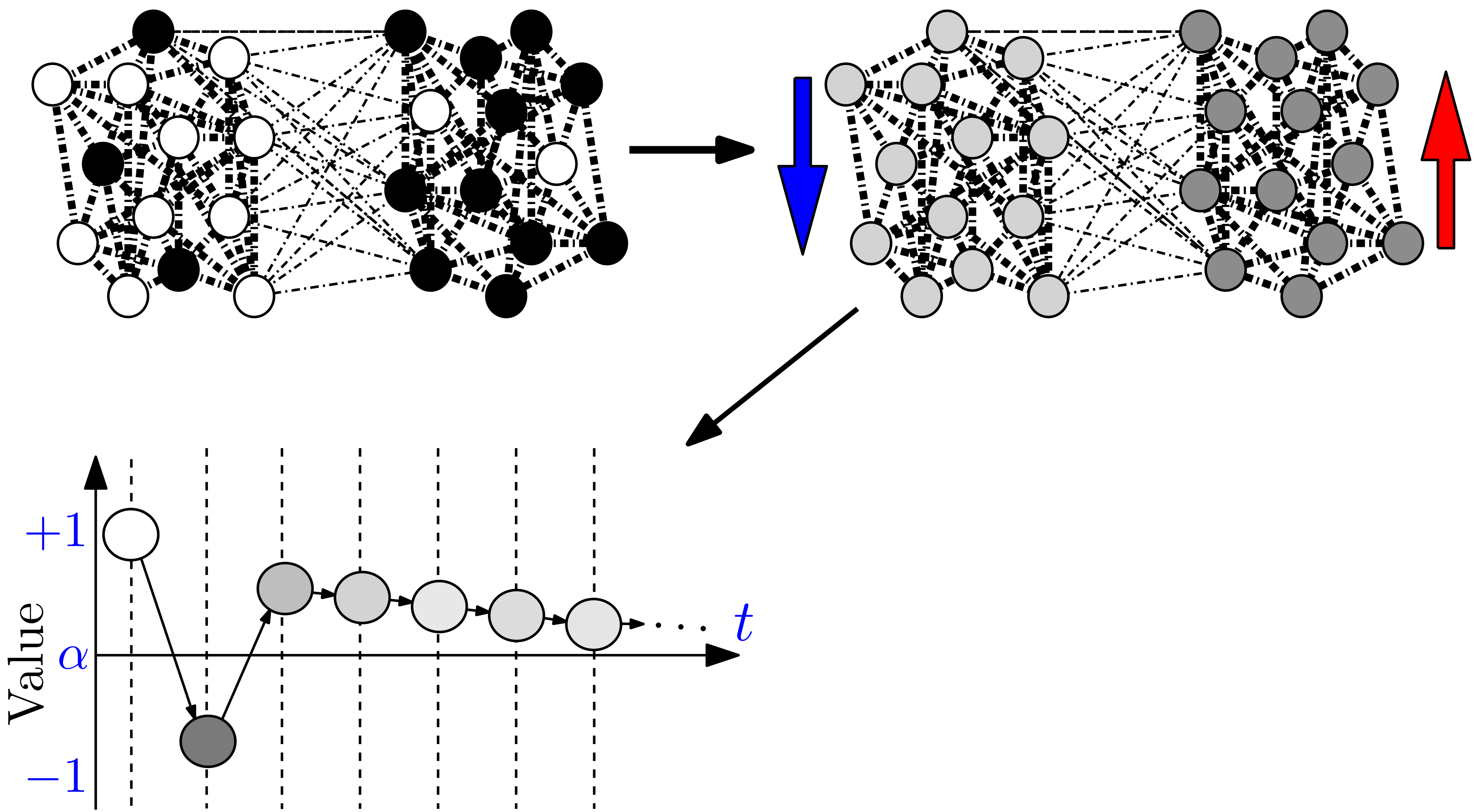

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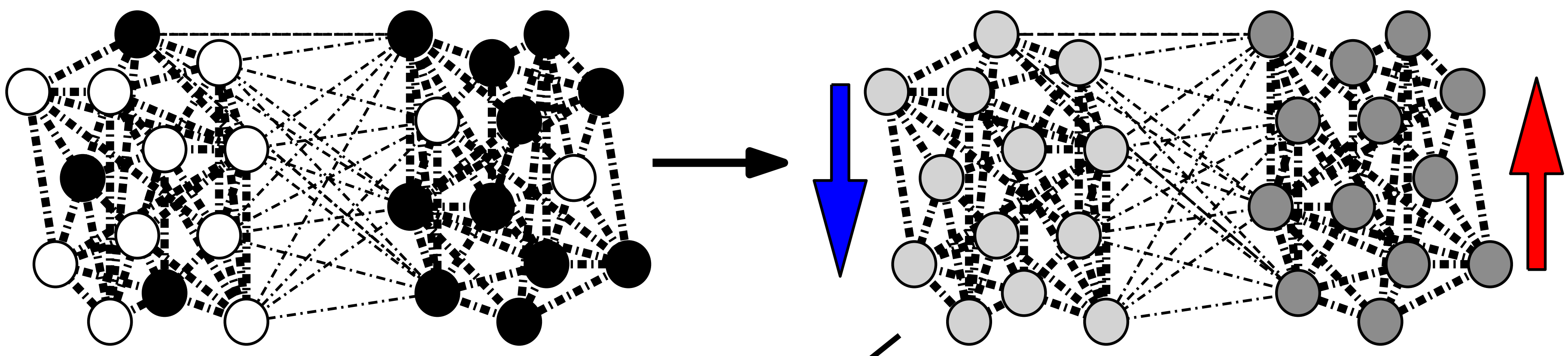
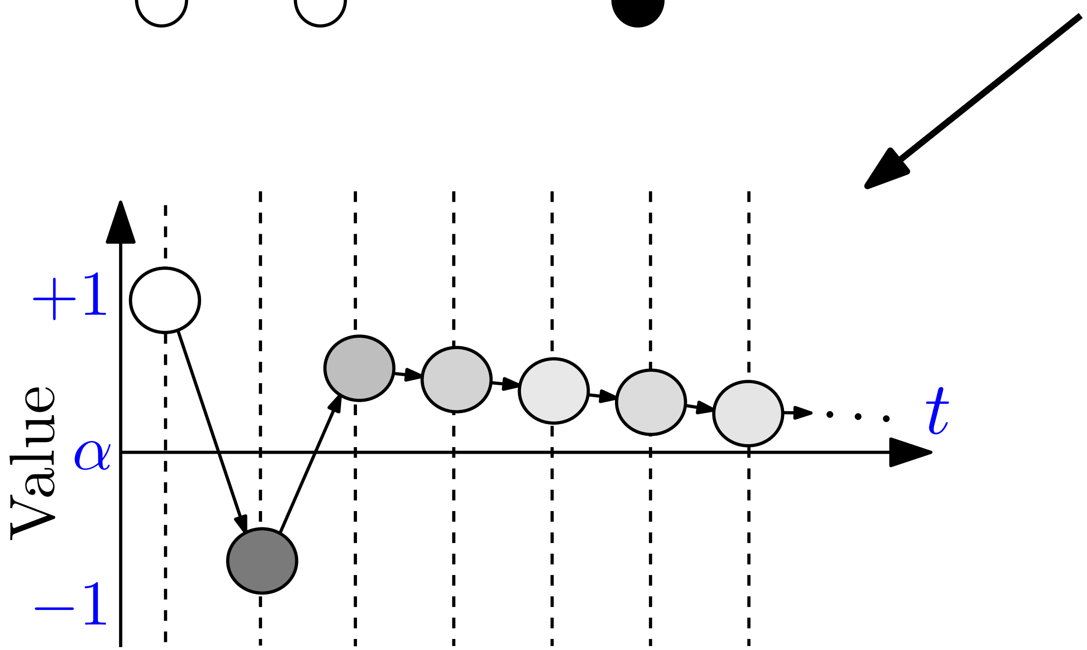




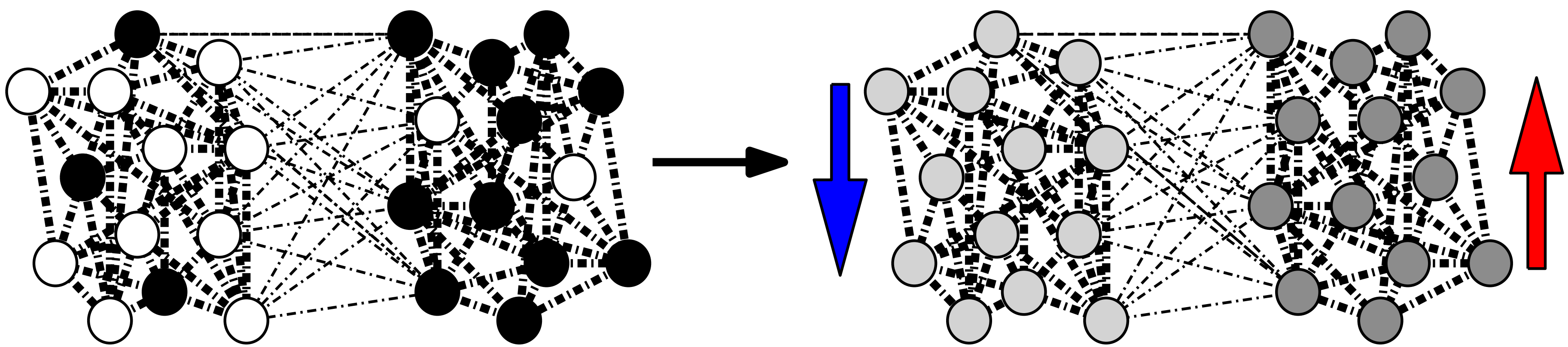
$v_1, \ldots, v_n$ eigenvectors of random walk matrix P:

$$v_1 = \mathbb{1} = (1, \ldots, 1)$$

$$v_2 \approx \chi = (1, \ldots, 1, -1, \ldots, -1)$$

“nice” graph

Community Detection via Averaging Dynamics



[SODA '17](Informal). $G = (V_1 \dot{\bigcup} V_2, E)$ s.t.
i) $\chi = \mathbf{1}_{V_1} - \mathbf{1}_{V_2}$ close to right-eigenvector of eigenvalue λ_2 of transition matrix of G, and
ii) gap between λ_2 and $\lambda = \max\{\lambda_3, |\lambda_n|\}$ sufficiently large,
then Averaging (approximately) identifies (V_1, V_2).

Thank you!

Analysis on Regular SBM

$$P \longrightarrow$$ symmetric $\implies$ orthonormal eigenvectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$ and real eigenvalues $\lambda_1, \ldots, \lambda_n$.

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$$\mathbf{v}_1 = \frac{1}{\sqrt{n}} \mathbf{1}$$

Regular SBM $\Longrightarrow$ $P\chi = \left(\frac{a-b}{a+b}\right) \cdot \chi$

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$$\frac{1}{a+b} \left(\begin{array}{c:c} \cdots\cdots\cdots & \cdots\cdots\cdots \\ \cdots a \text{ “1”s} \cdots & \cdots b \text{ “1”s} \cdots \\ \cdots\cdots\cdots & \cdots\cdots\cdots \\ \hdashline \cdots\cdots\cdots & \cdots\cdots\cdots \\ \cdots b \text{ “1”s} \cdots & \cdots a \text{ “1”s} \cdots \\ \cdots\cdots\cdots & \cdots\cdots\cdots \end{array} \right) \cdot \begin{pmatrix} 1 \\ \vdots \\ 1 \\ -1 \\ \vdots \\ -1 \end{pmatrix} = \frac{a-b}{a+b} \begin{pmatrix} 1 \\ \vdots \\ 1 \\ -1 \\ \vdots \\ -1 \end{pmatrix}$$

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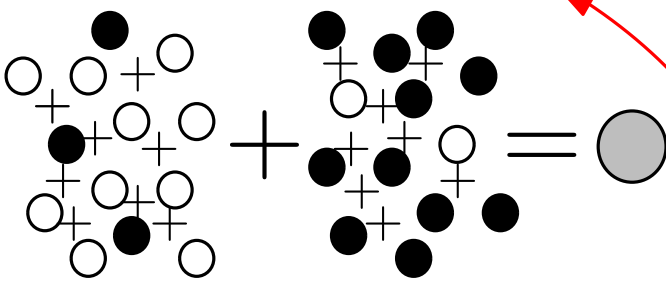
W.h.p. $\max\{\lambda_3, |\lambda_n|\}(1+\delta) < \frac{a-b}{a+b} = \lambda_2$, then

$$\mathbf{x}^{(t+1)} = \frac{1}{n}(\mathbf{1}^\intercal \mathbf{x}^{(0)})\mathbf{1} + \lambda_2^t \frac{1}{n}(\chi^\intercal \mathbf{x}^{(0)})\chi + \mathbf{e}^{(t)}$$

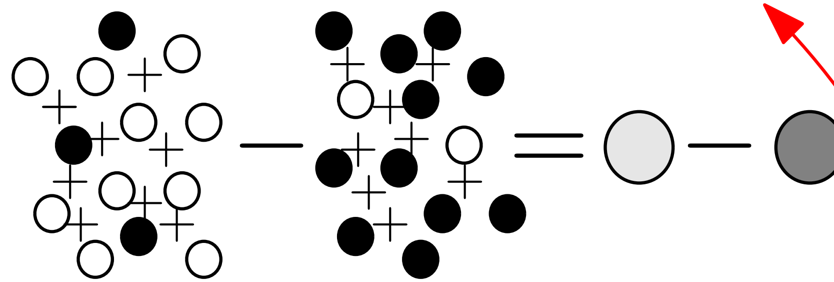
with $\|\mathbf{e}^{(t)}\| \leq (\max\{\lambda_3, |\lambda_n|\})^t \sqrt{n}$

Analysis on Regular SBM

$\frac{1}{n}\sum_{u \in V} \mathbf{x}^{(0)}(u)$



$\left(\frac{1}{n}\sum_{u \in V_1} \mathbf{x}^{(0)}(u) - \frac{1}{n}\sum_{u \in V_2} \mathbf{x}^{(0)}(u)\right)$



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$$\mathbf{x}^{(t)} = \frac{1}{n}(\mathbf{1}^\intercal \mathbf{x}^{(0)})\mathbf{1} + \lambda_2^t \frac{1}{n}(\chi^\intercal \mathbf{x}^{(0)})\chi + \mathbf{e}^{(t)}$$

$$\mathbf{x}^{(t)} - \mathbf{x}^{(t-1)} = (\chi^\intercal \mathbf{x}^{(0)})\lambda_2^{t-1}(\lambda_2 - 1)\chi + \underbrace{\mathbf{e}^{(t)} - \mathbf{e}^{(t-1)}}_{\ll \lambda_2^{t-1} \text{ if } t = \Omega(\log n)}$$

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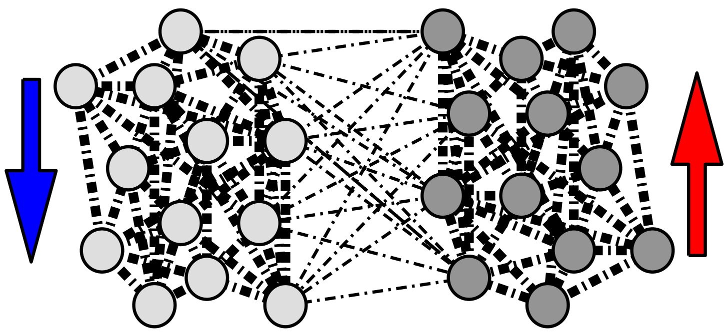

$$\boxed{\mathrm{sign}(\mathbf{x}^{(t)}(u) - \mathbf{x}^{(t-1)}(u)) \propto \mathrm{sign}(\chi(u))}$$