Noisy Rumor Spreading and Plurality Consensus

Emanuele Natale†

joint work with

Pierre Fraigniaud*

* CNRS, IRIF — Institut de Recherche en Informatique Fondamentale

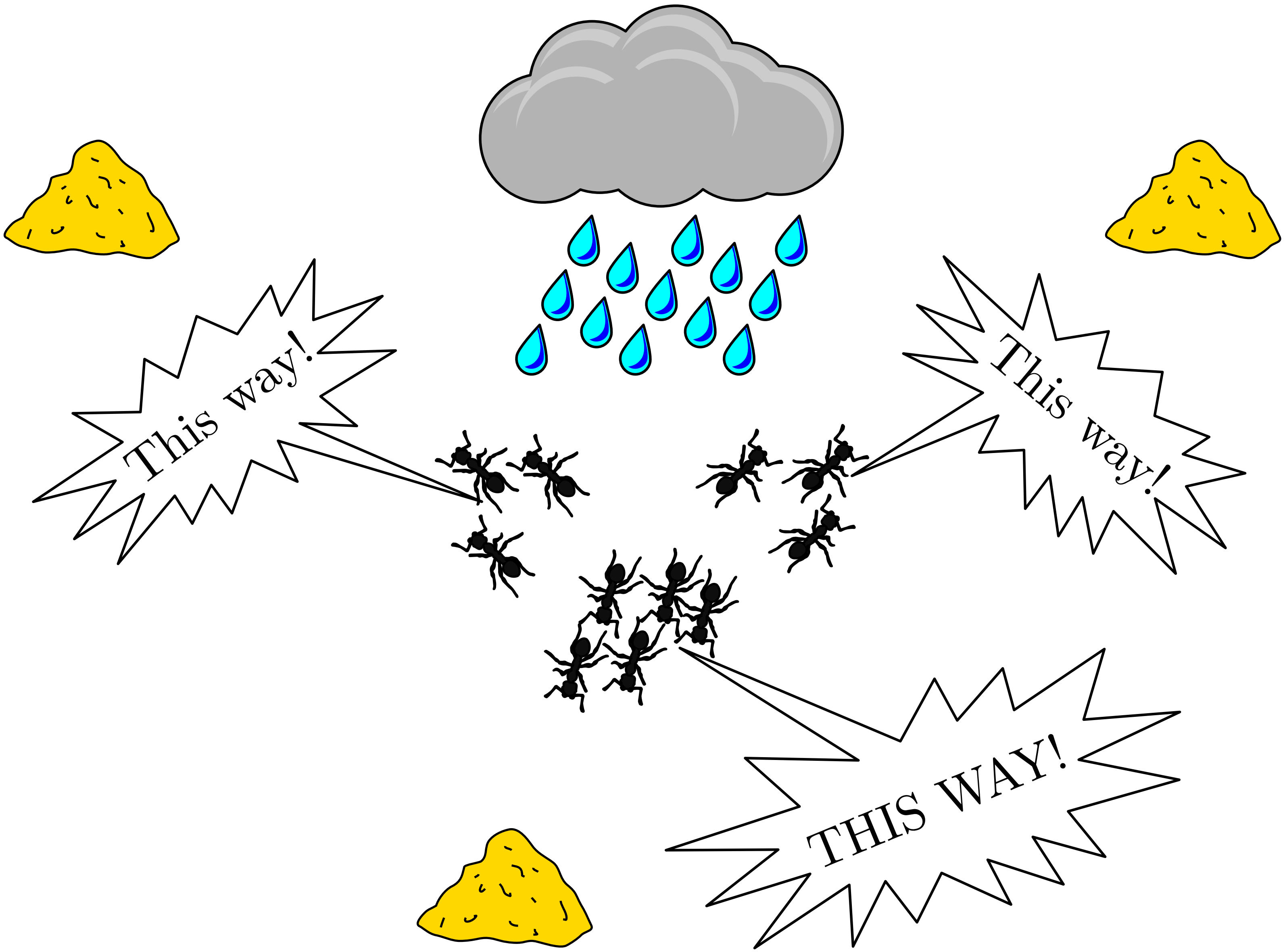
† Sapienza Università di Roma

ACM Symposium on
Principles of Distributed Computing
July 25-29, 2016
Chicago, Illinois

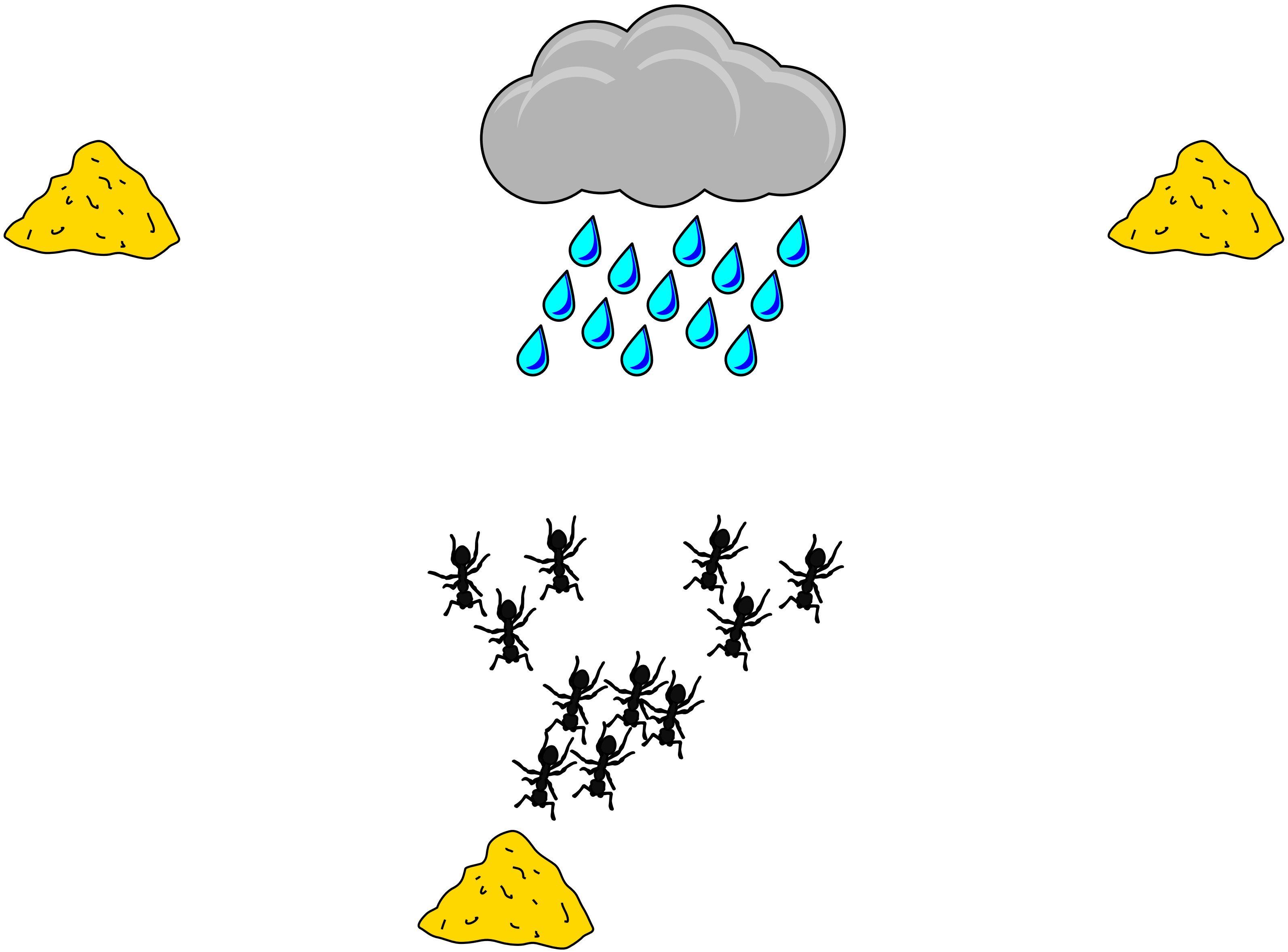
Rumor-Spreading Problem

Rumor-Spreading Problem

Plurality Consensus Problem

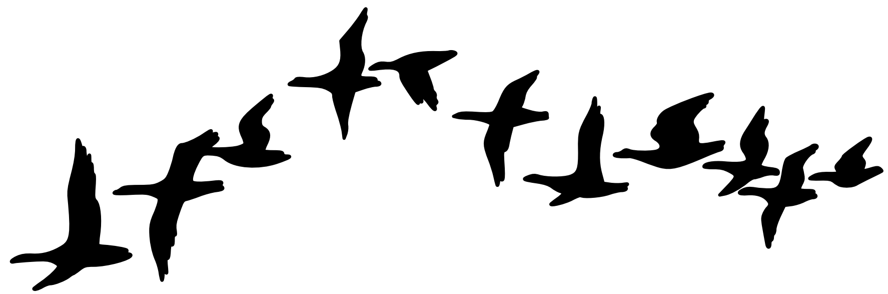


Plurality Consensus Problem



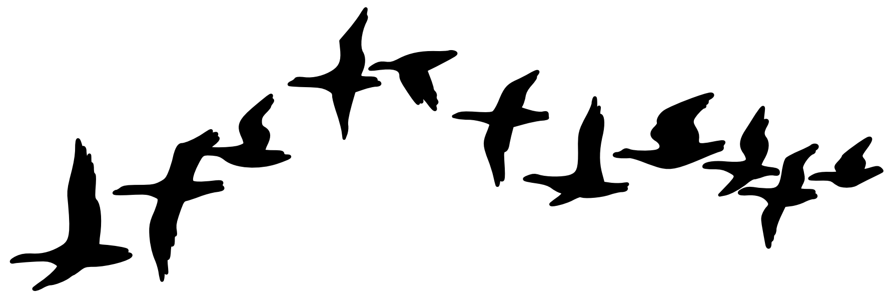
Some examples (Plurality Consensus)

Flocks of birds [Ben-Shahar et al. '10]

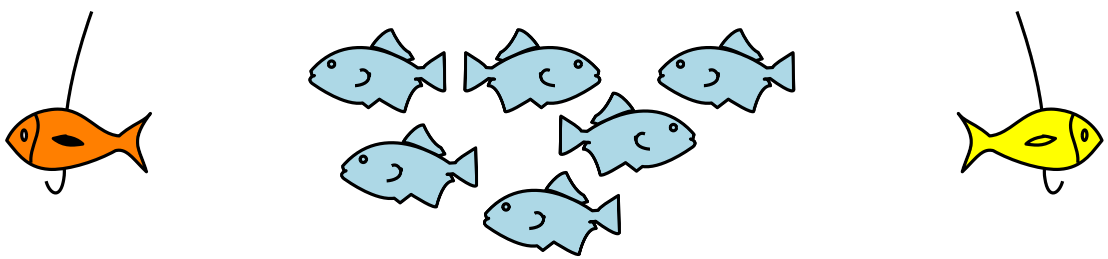


Some examples (Plurality Consensus)

Flocks of birds [Ben-Shahar et al. '10]

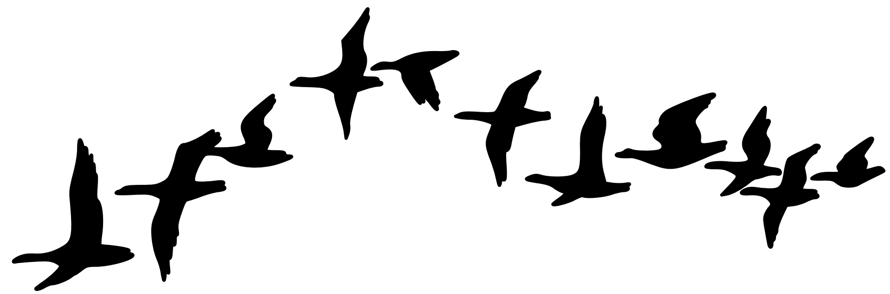


Schools of fish [Sumpter et al. '08]

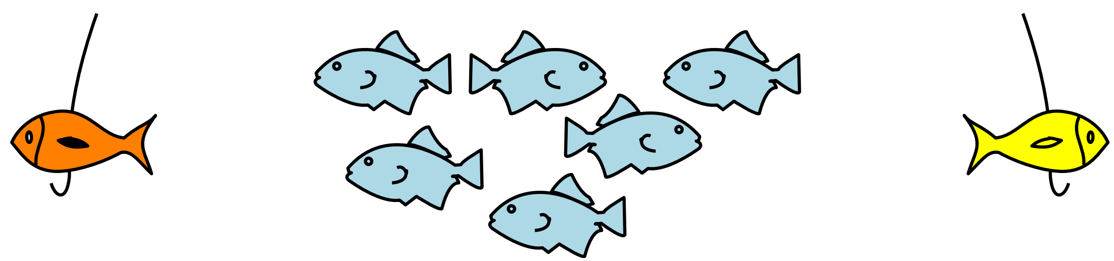


Some examples (Plurality Consensus)

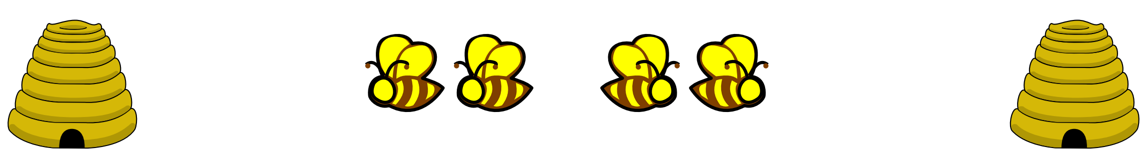
Flocks of birds [Ben-Shahar et al. '10]



Schools of fish [Sumpter et al. '08]

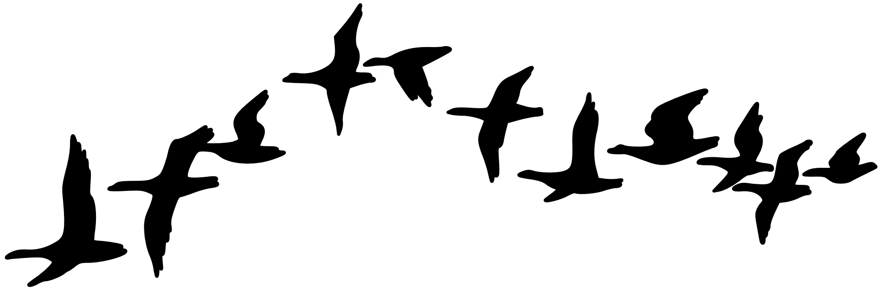


Insects colonies [Franks et al. '02]

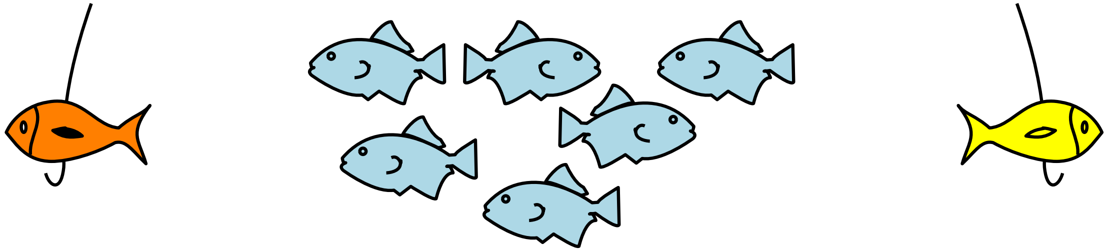


Some examples (Plurality Consensus)

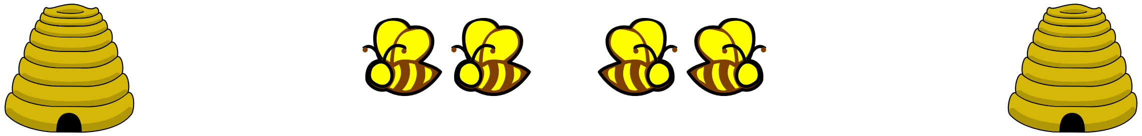
Flocks of birds [Ben-Shahar et al. '10]



Schools of fish [Sumpter et al. '08]



Insects colonies [Franks et al. '02]



Eukaryotic cells [Cardelli et al. '12]

Animal Communication Despite Noise

Noise affects animal communication,
but animals cannot use *coding theory*...

Animal Communication Despite Noise

Noise affects animal communication,
but animals cannot use *coding theory*...

O. Feinerman, B. Haeupler and A. Korman.
Breathe before speaking: efficient information dissemination despite noisy, limited and anonymous communication. (PODC '14)

$\Longrightarrow$ Natural rules efficiently solve rumor spreading and plurality consensus despite noise.

Animal Communication Despite Noise

Noise affects animal communication,
but animals cannot use *coding theory*...

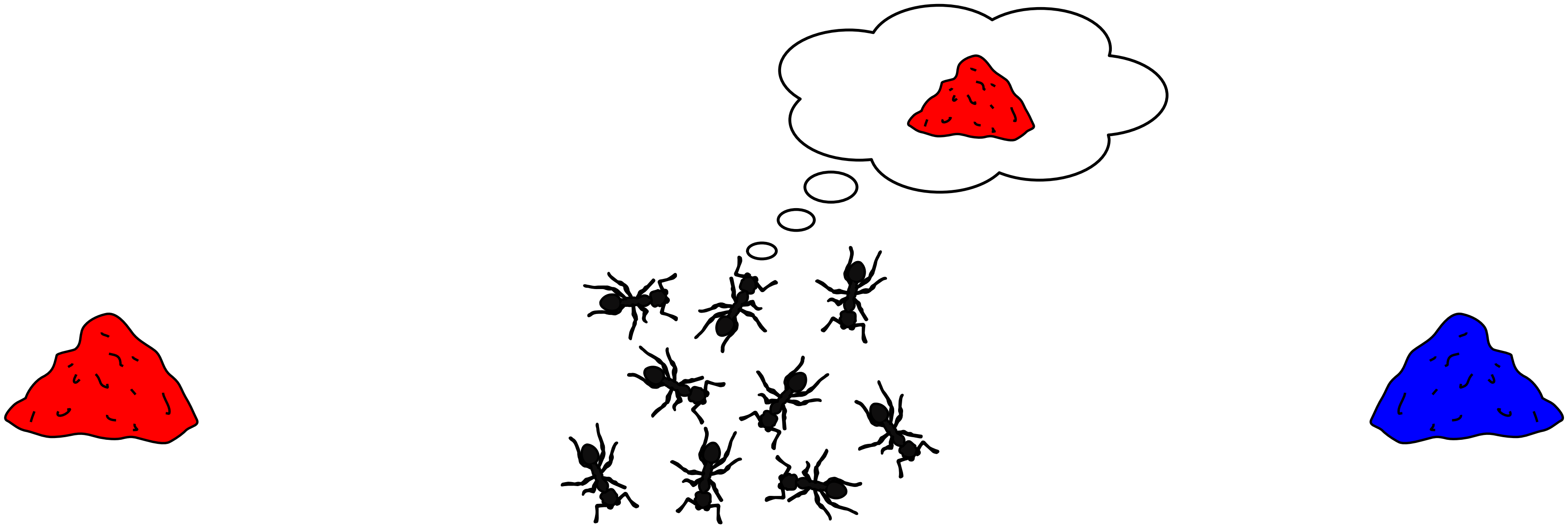
O. Feinerman, B. Haeupler and A. Korman.
Breathe before speaking: efficient information dissemination despite noisy, limited and anonymous communication. (PODC '14)

$\Longrightarrow$ Natural rules efficiently solve rumor spreading and plurality consensus despite noise.

They only consider the binary-opinion case.
Our contribution: generalize to **many opinions**.

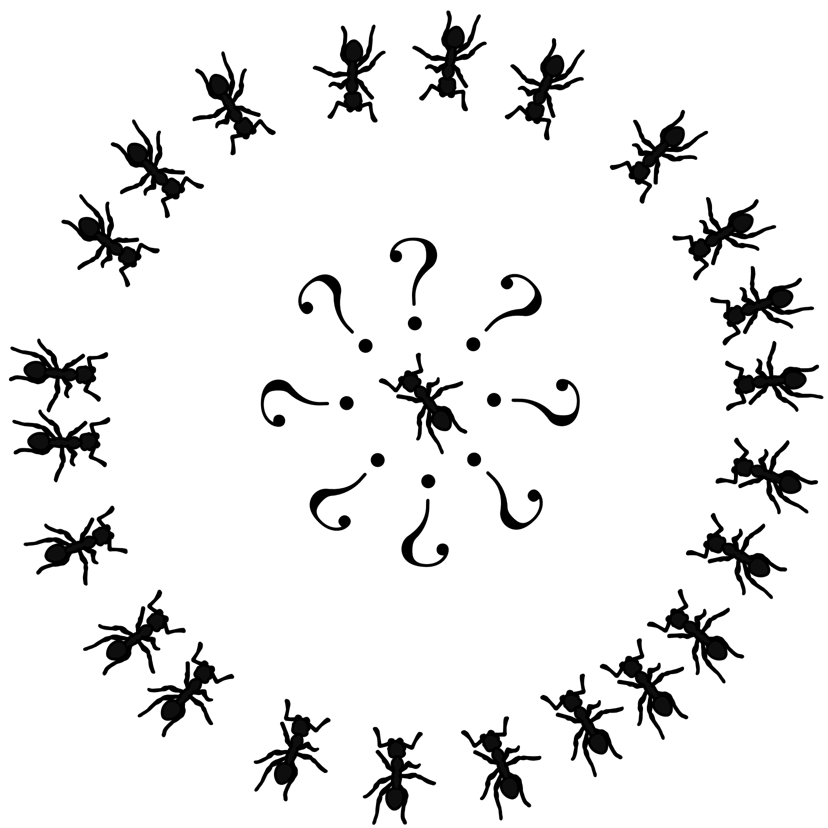
Binary Case - Model

n agents. One agent has one bit to spread.



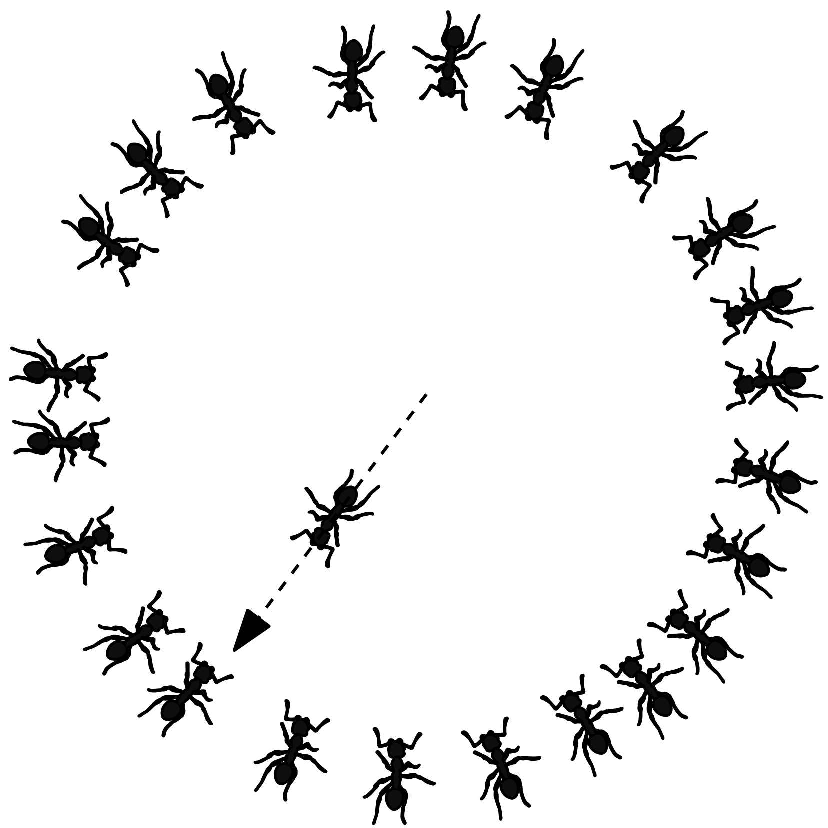
Binary Case - Model

Communication model: $\mathcal{PUSH}$ model [Pittel '87]: at each round each agent can send a bit to another one chosen uniformly at random.



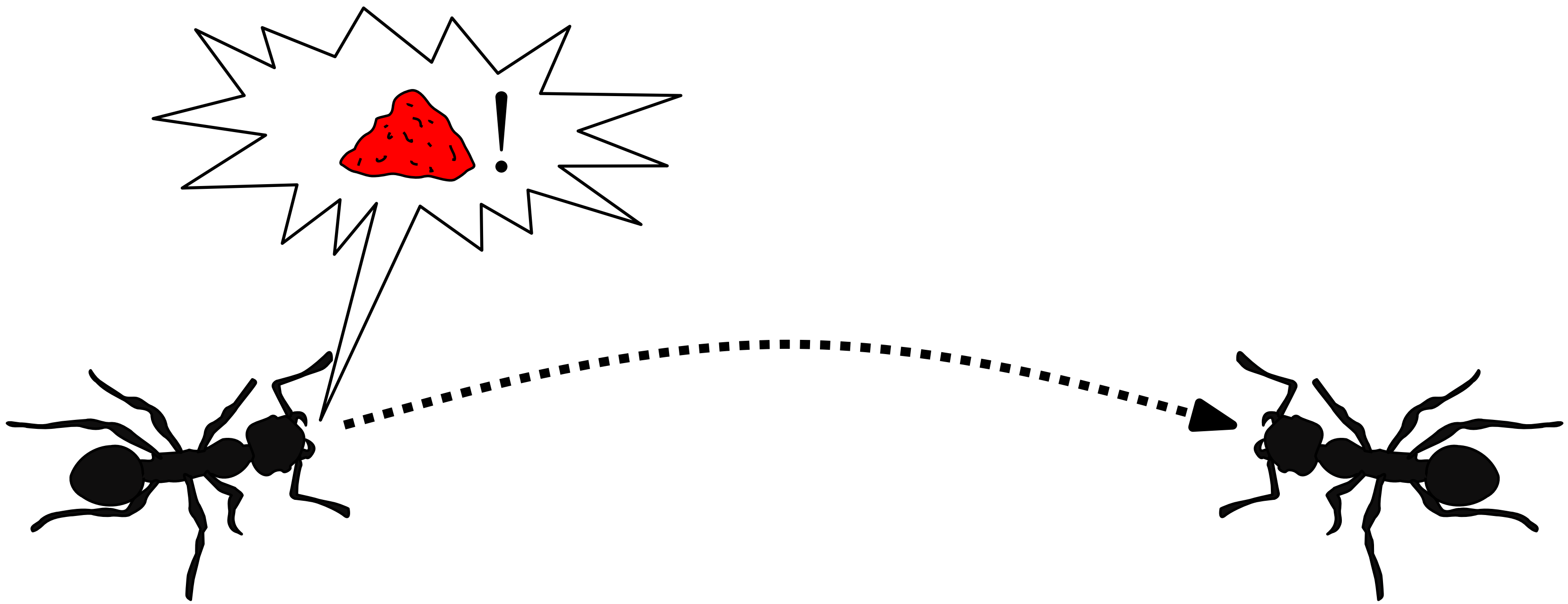
Binary Case - Model

Communication model: $\mathcal{PUSH}$ model [Pittel '87]: at each round each agent can send a bit to another one chosen uniformly at random.



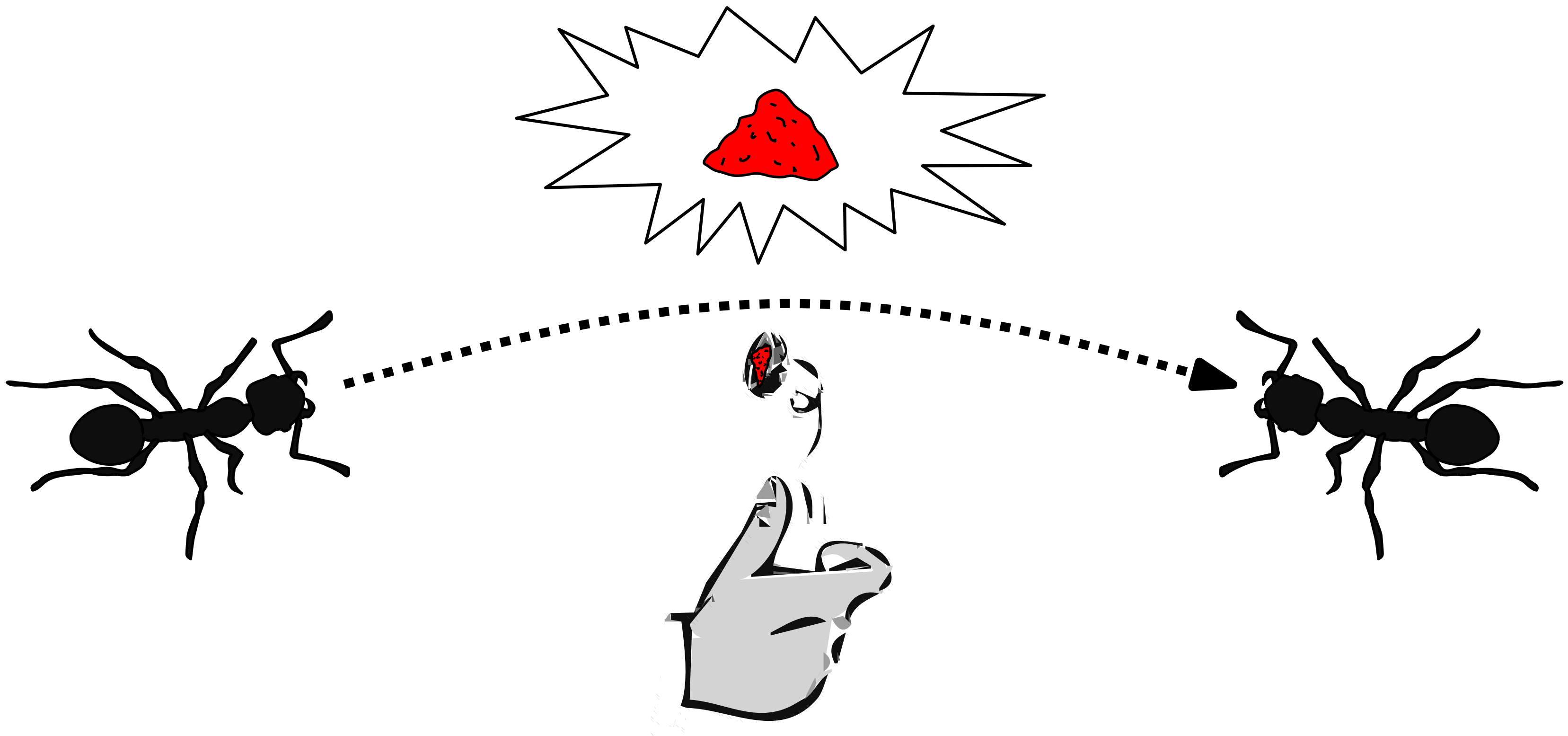
Binary Case - Model

Noise: before being received, each bit is flipped with probability $1/2 - \epsilon$ ($\epsilon = n^{-const}$).



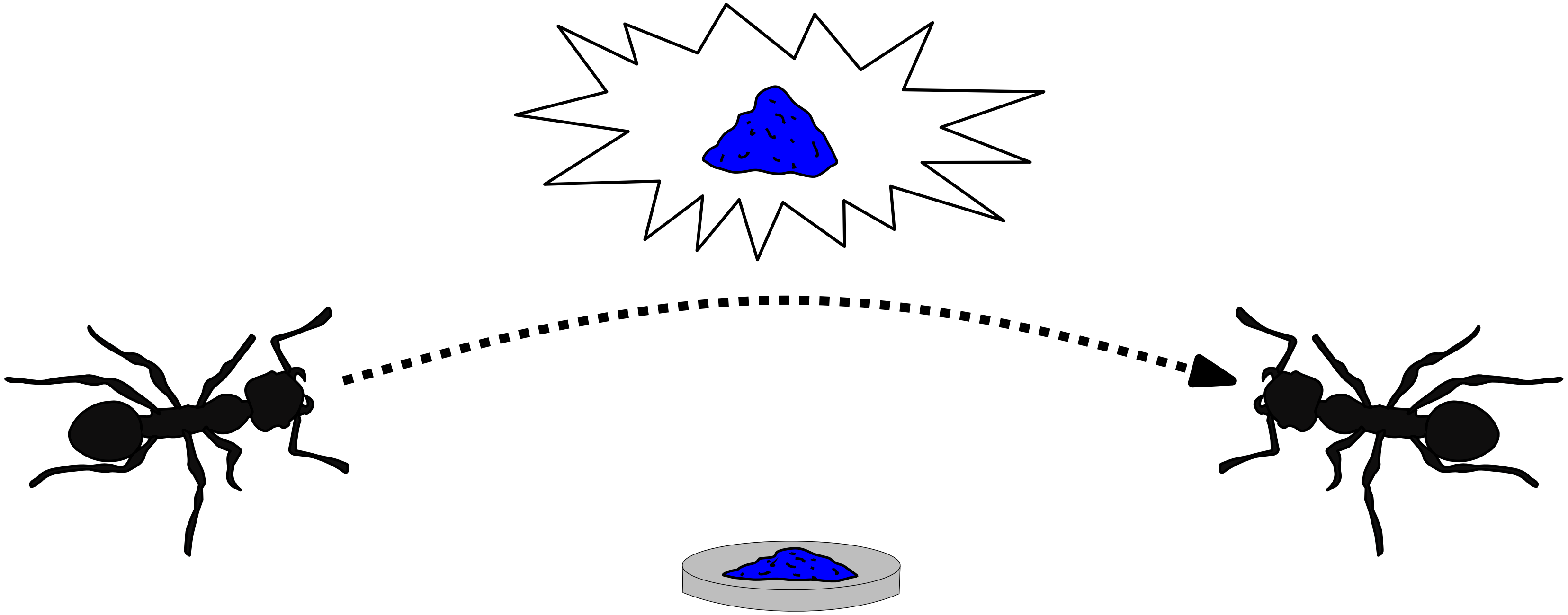
Binary Case - Model

Noise: before being received, each bit is flipped with probability $1/2 - \epsilon$ ($\epsilon = n^{-const}$).



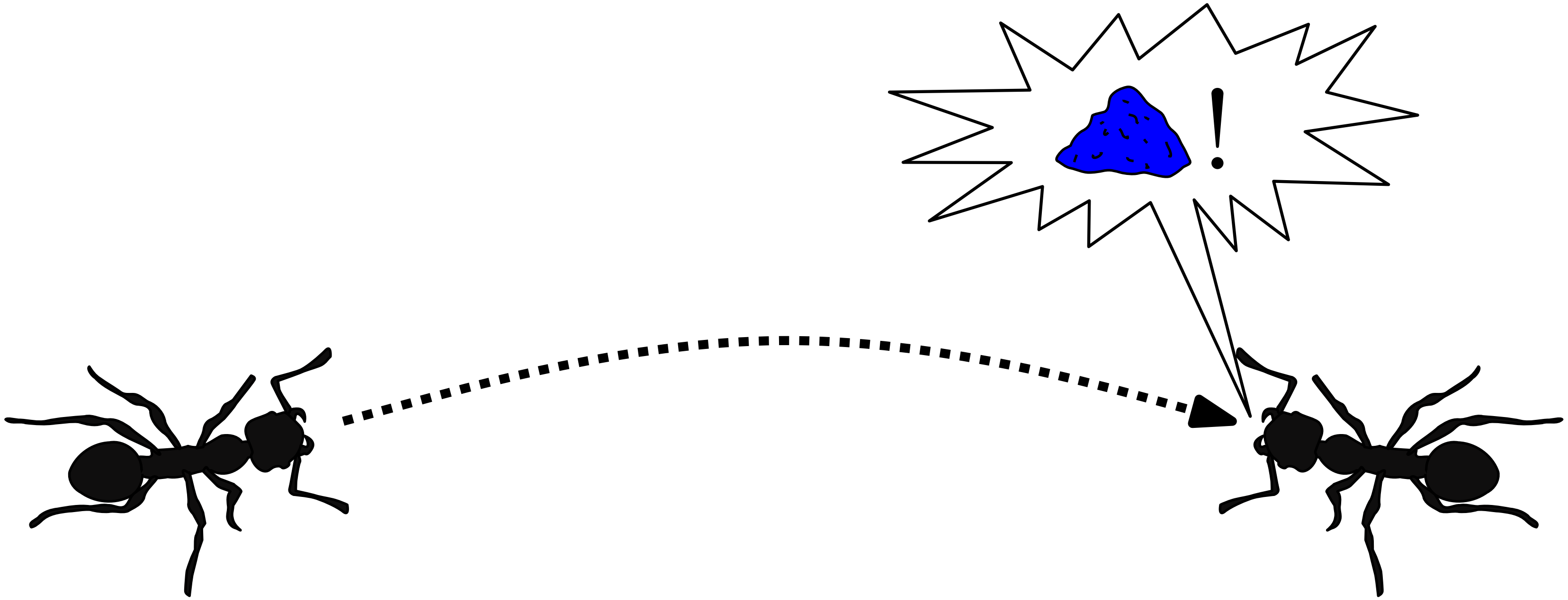
Binary Case - Model

Noise: before being received, each bit is flipped with probability $1/2 - \epsilon$ ($\epsilon = n^{-const}$).

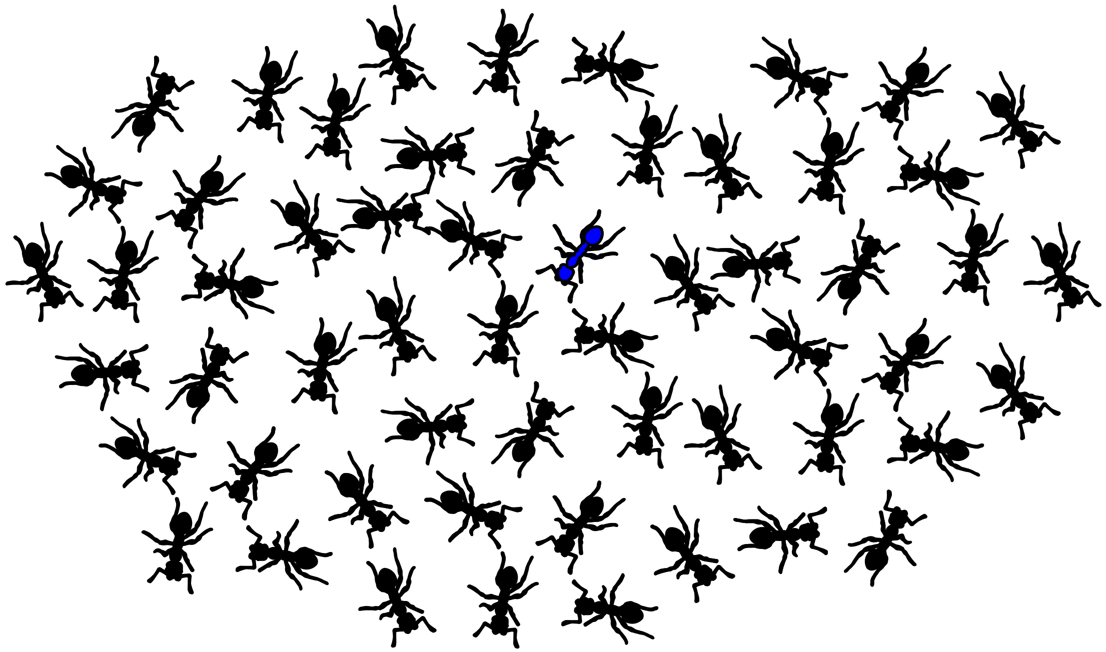


Binary Case - Model

Noise: before being received, each bit is flipped with probability $1/2 - \epsilon$ ($\epsilon = n^{-const}$).



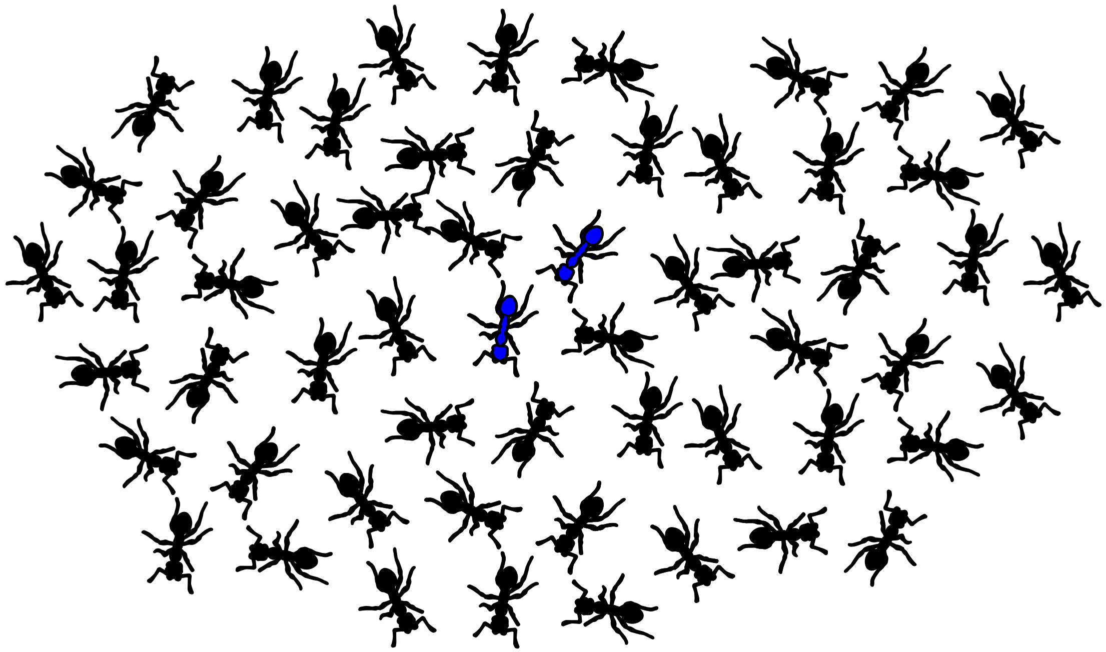
Breathe Before Speaking



trivial strategy

blue vs red:
1/0

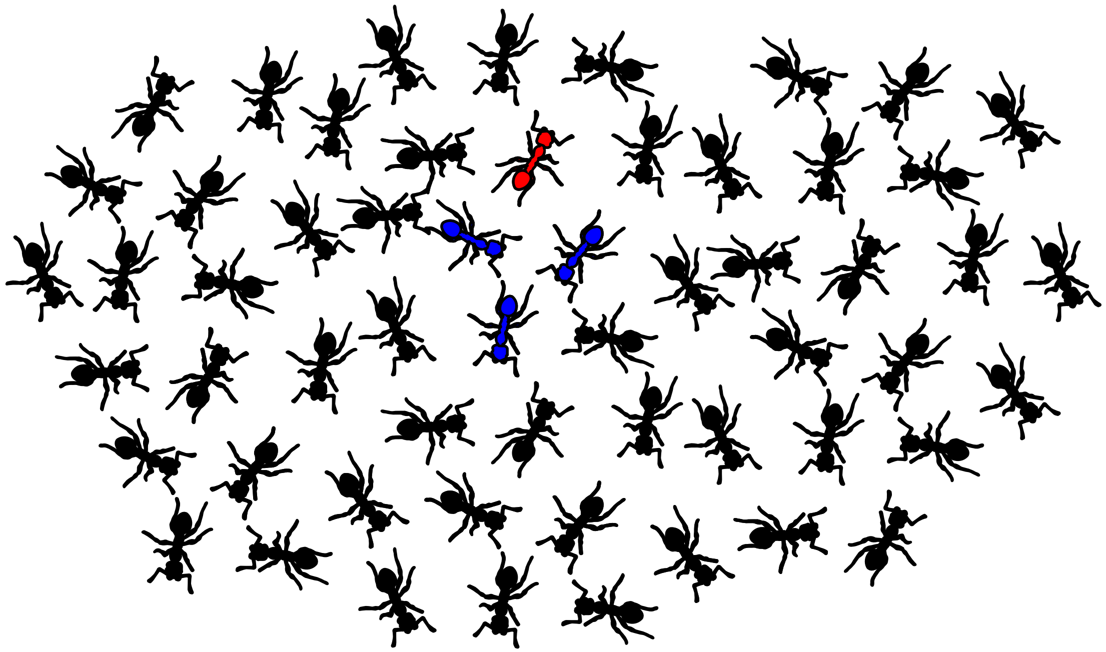
Breathe Before Speaking



trivial
strategy

blue vs red:
2/0

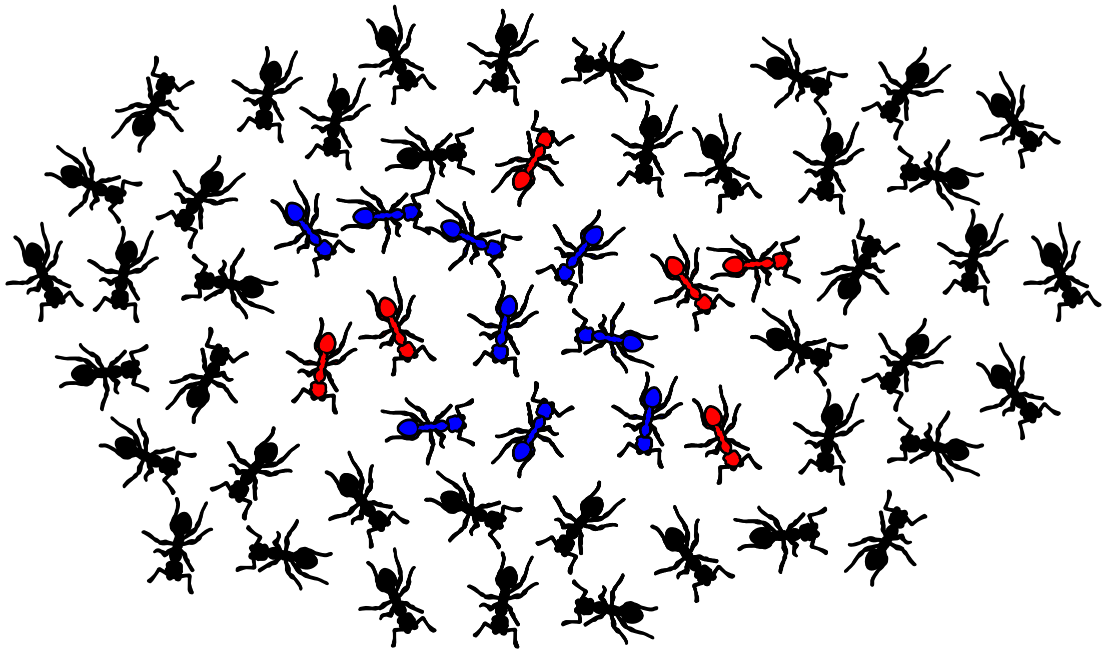
Breathe Before Speaking



trivial
strategy

blue vs red:
3/1

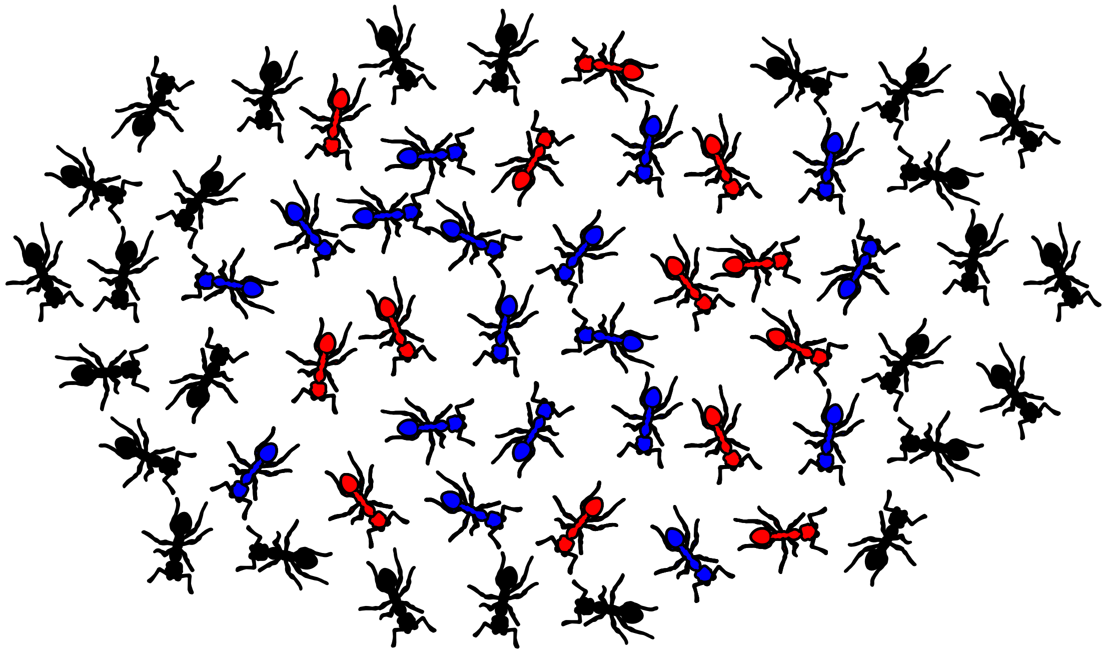
Breathe Before Speaking



trivial strategy

blue vs red:
$9/6 = 1.5$

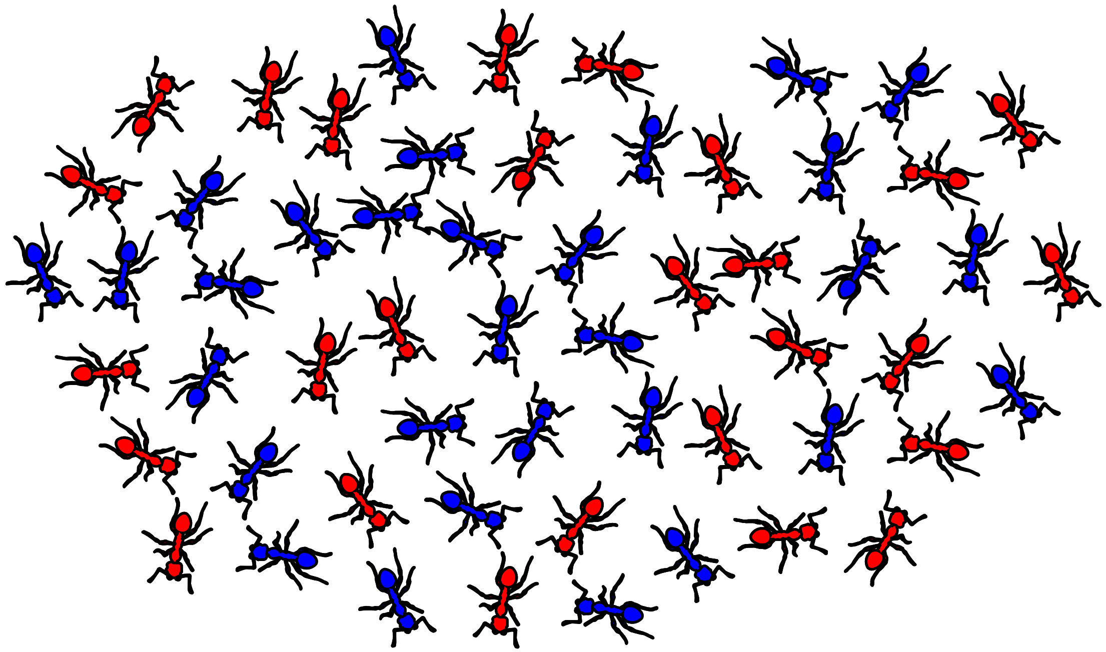
Breathe Before Speaking



trivial
strategy

blue vs red:
18/13 ≈ 1.4

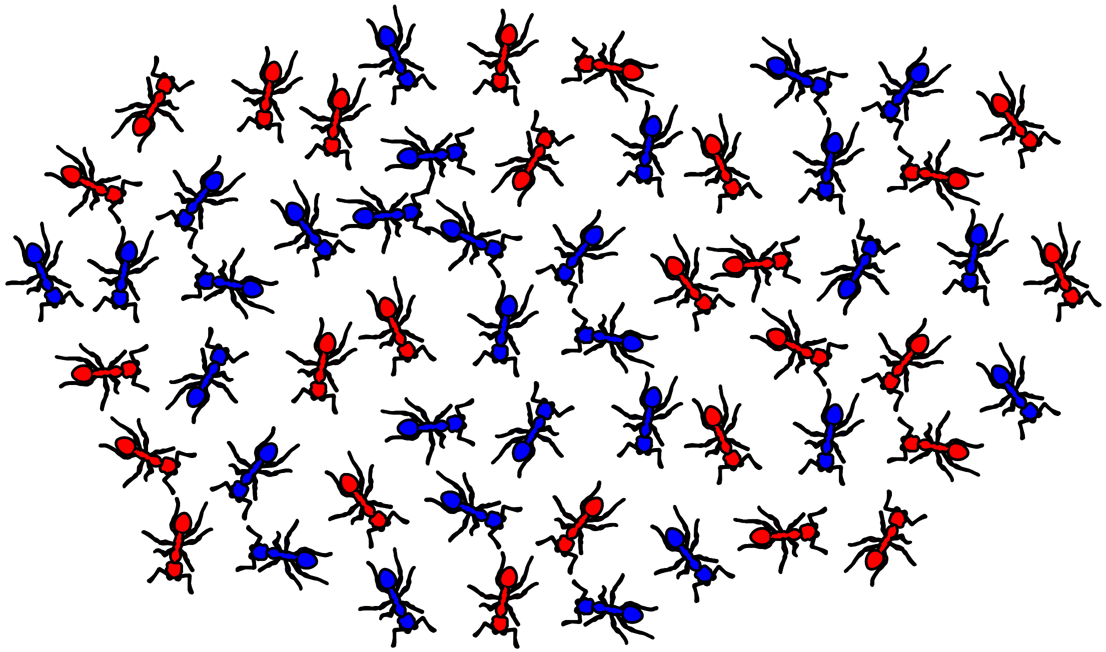
Breathe Before Speaking



trivial strategy

blue vs red:
$35/29 \approx 1.2$

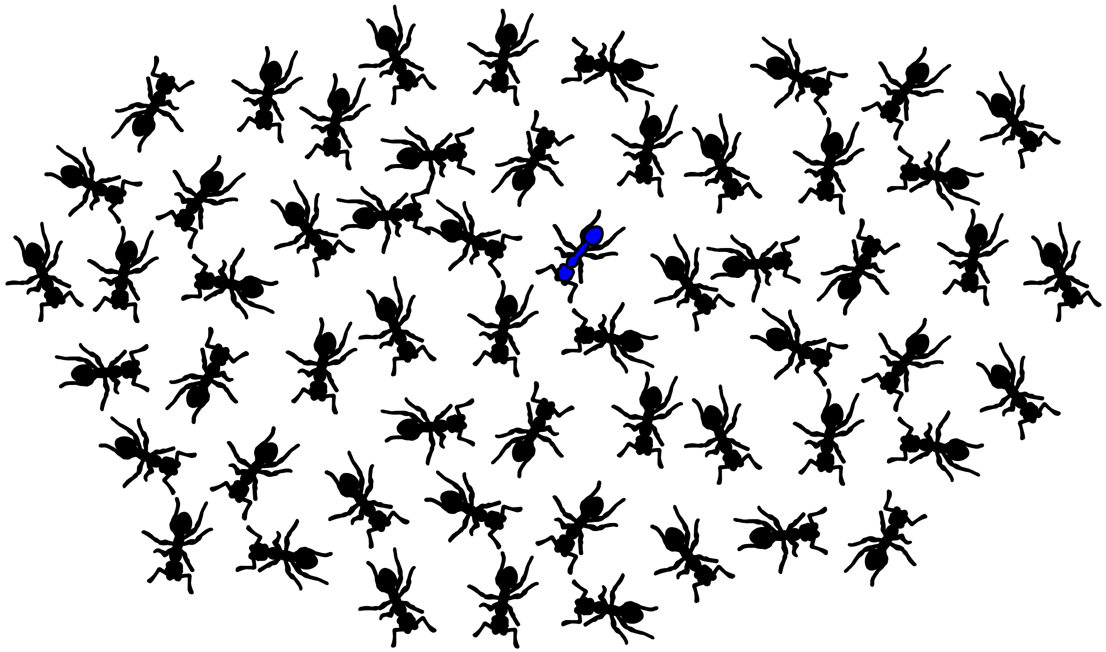
Breathe Before Speaking



blue vs red:

$35/29 \approx 1.2$

Breathe Before Speaking



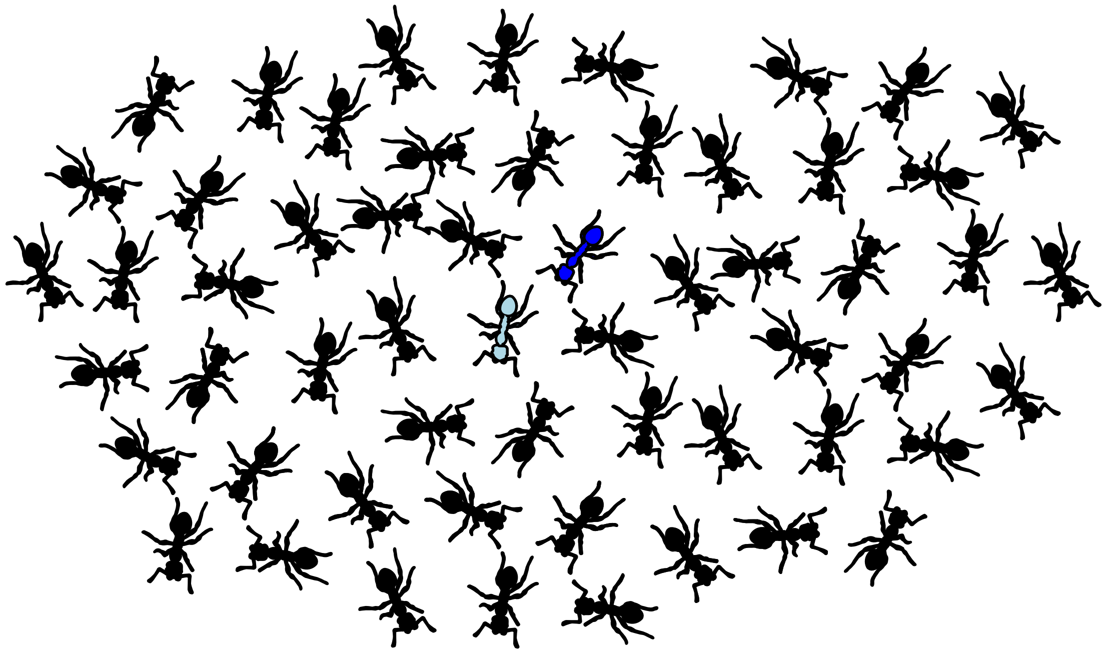
Stage 1: Spreading

blue vs red:
1/0

"[...] ants effectively self-restrict their own tendency to engage in further interactions that would excite further nest-mates."

(Razin et al. '13)

Breathe Before Speaking



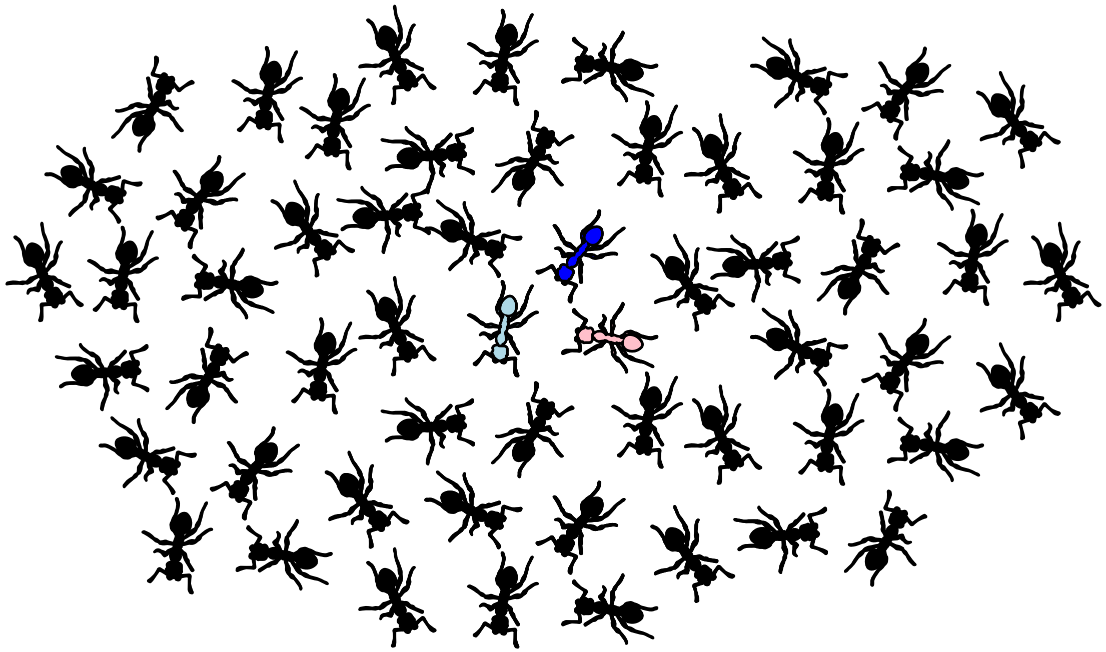
Stage 1: Spreading

blue vs red:
1/0

"[...] ants effectively self-restrict their own tendency to engage in further interactions that would excite further nest-mates."

(Razin et al. '13)

Breathe Before Speaking



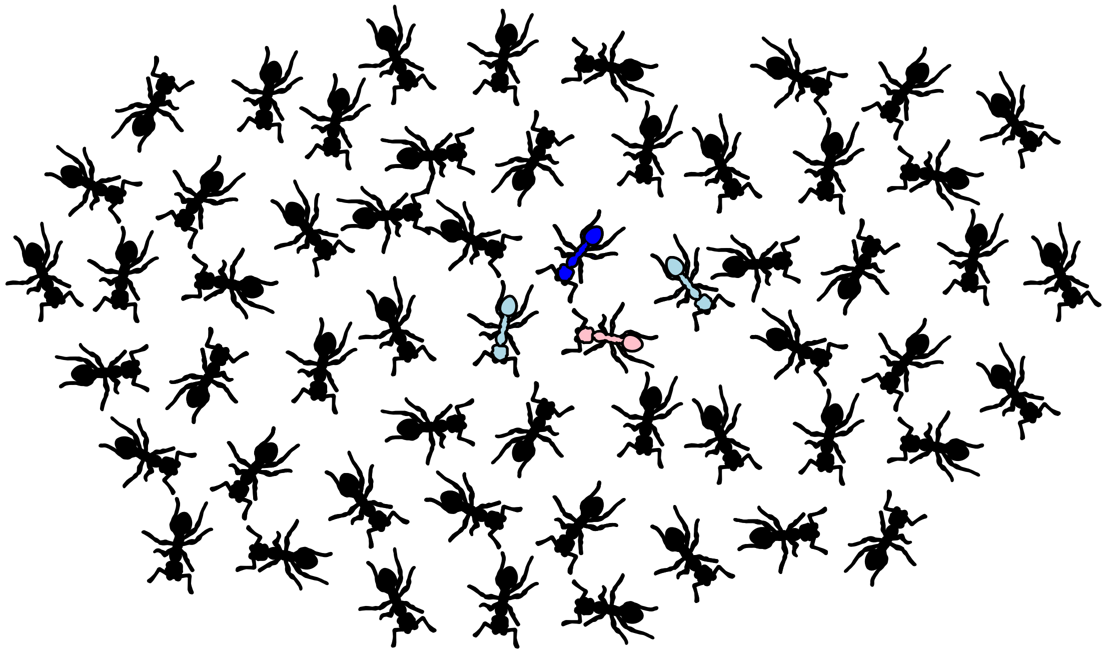
Stage 1: Spreading

blue vs red:
1/0

"[...] ants effectively self-restrict their own tendency to engage in further interactions that would excite further nest-mates."

(Razin et al. '13)

Breathe Before Speaking



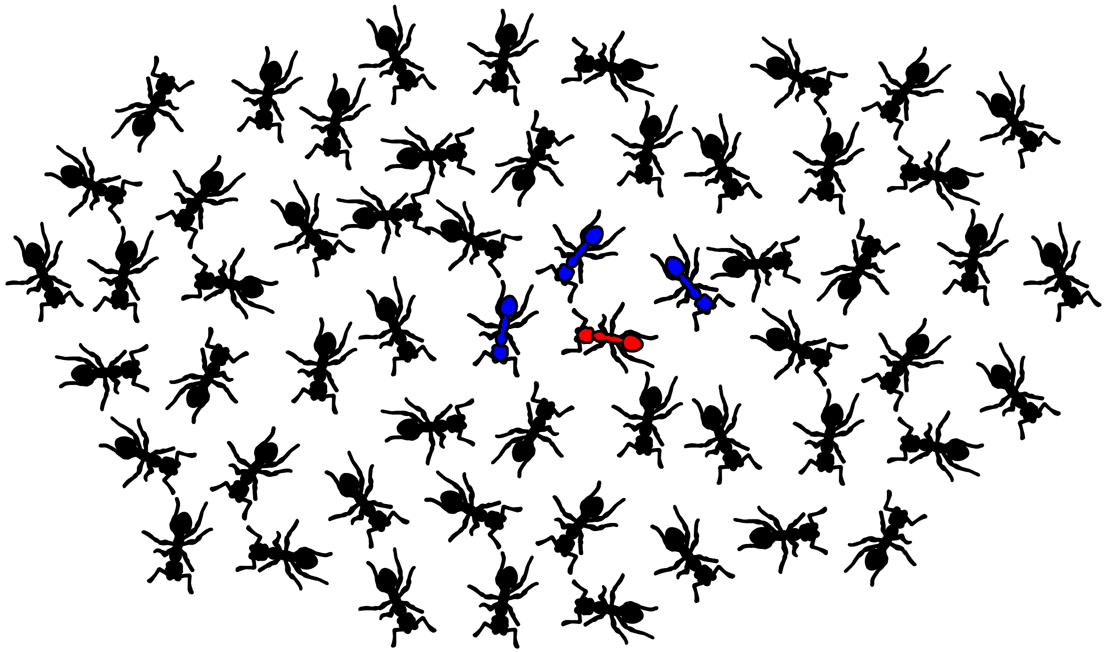
Stage 1: Spreading

blue vs red:
1/0

"[...] ants effectively self-restrict their own tendency to engage in further interactions that would excite further nest-mates."

(Razin et al. '13)

Breathe Before Speaking



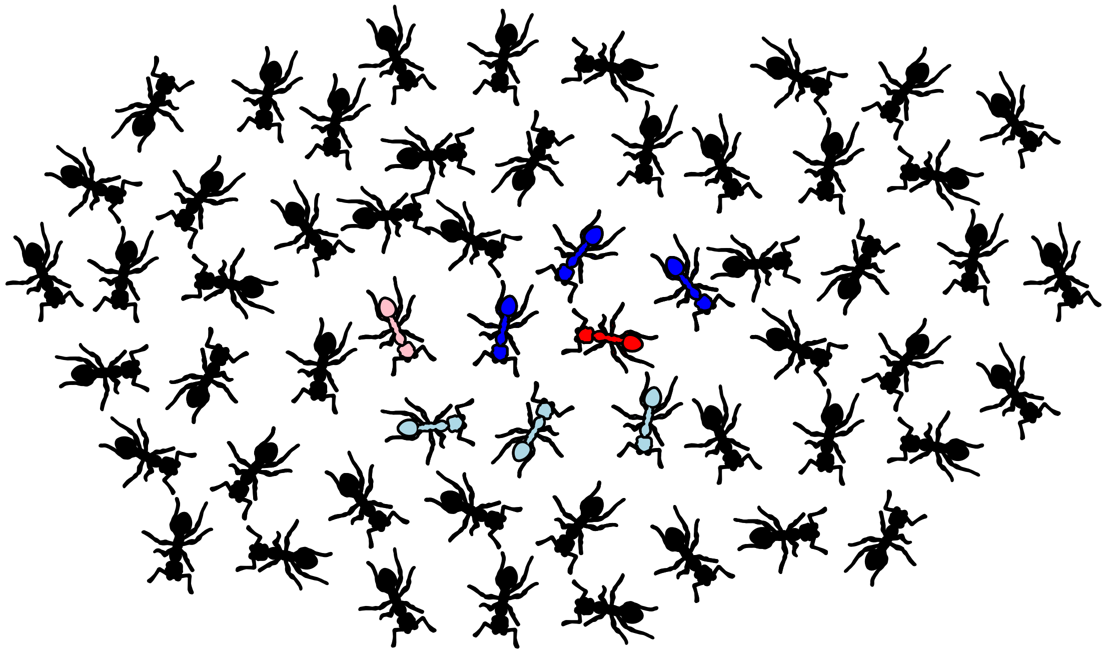
Stage 1: Spreading

blue vs red:
3/1

“[...] ants effectively self-restrict their own tendency to engage in further interactions that would excite further nest-mates.”

(Razin et al. ’13)

Breathe Before Speaking



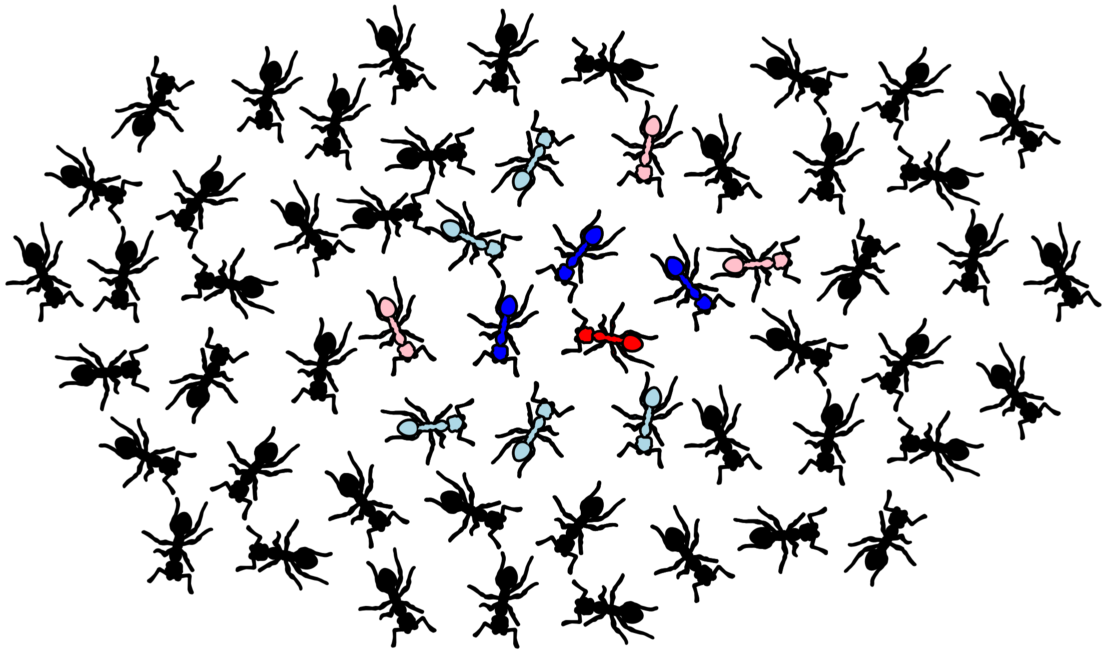
Stage 1: Spreading

blue vs red:
3/1

"[...] ants effectively self-restrict their own tendency to engage in further interactions that would excite further nest-mates."

(Razin et al. '13)

Breathe Before Speaking



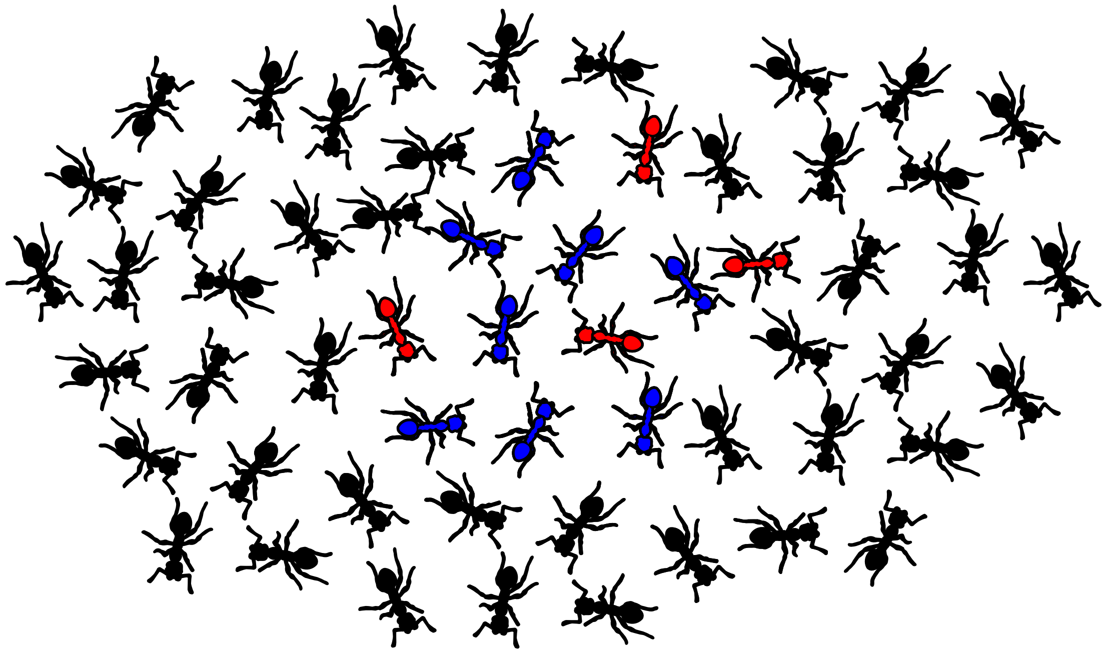
Stage 1: Spreading

blue vs red:
3/1

"[...] ants effectively self-restrict their own tendency to engage in further interactions that would excite further nest-mates."

(Razin et al. '13)

Breathe Before Speaking



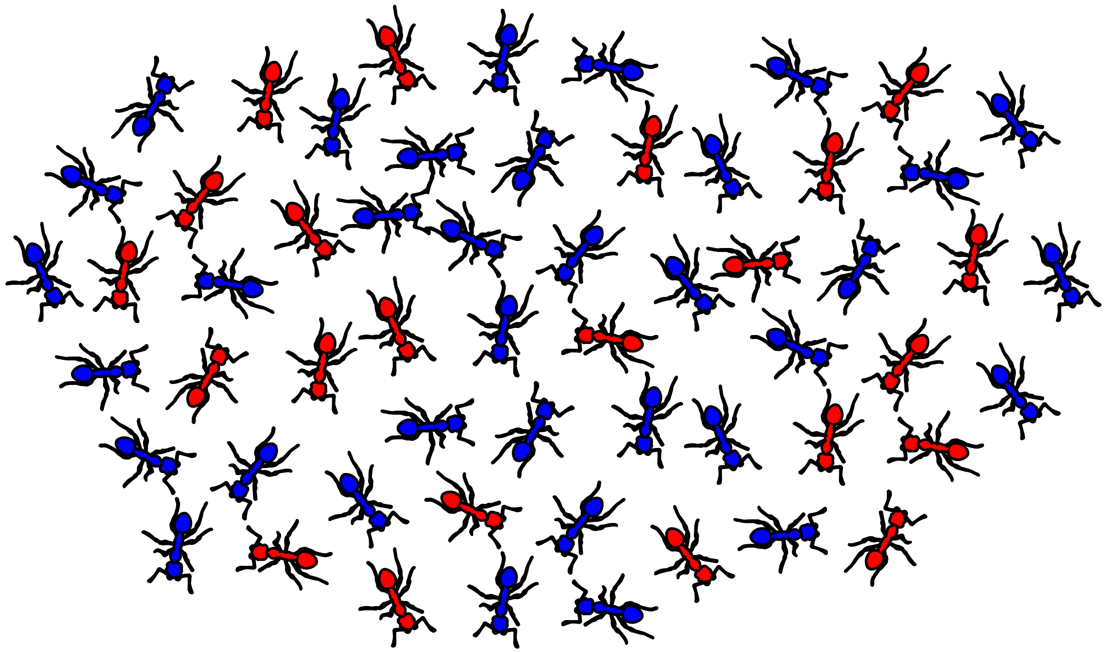
Stage 1: Spreading

blue vs red:
8/4

"[...] ants effectively self-restrict their own tendency to engage in further interactions that would excite further nest-mates."

(Razin et al. '13)

Breathe Before Speaking



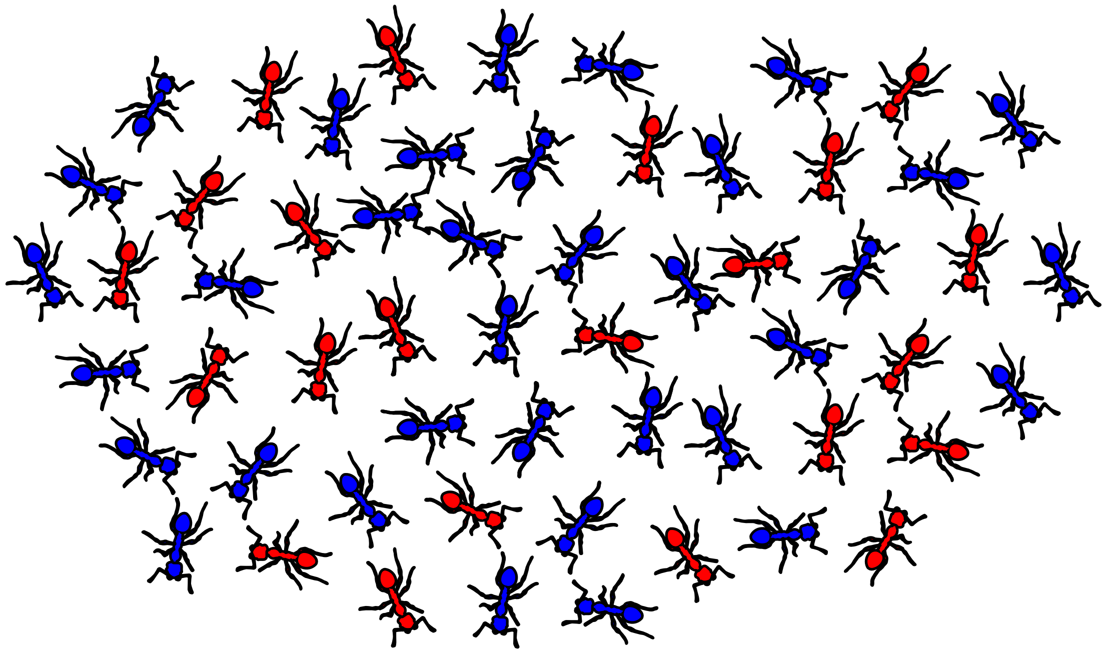
Stage 1: Spreading

blue vs red:
40/24 $\approx$ 1.7

"[...] ants effectively self-restrict their own tendency to engage in further interactions that would excite further nest-mates."

(Razin et al. '13)

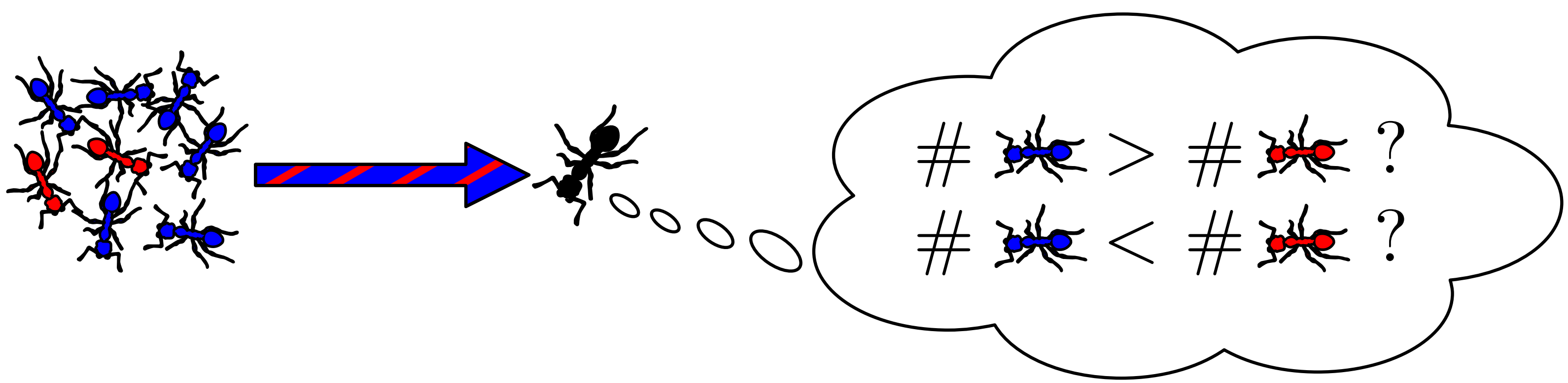
Breathe Before Speaking



Stage 1: Spreading

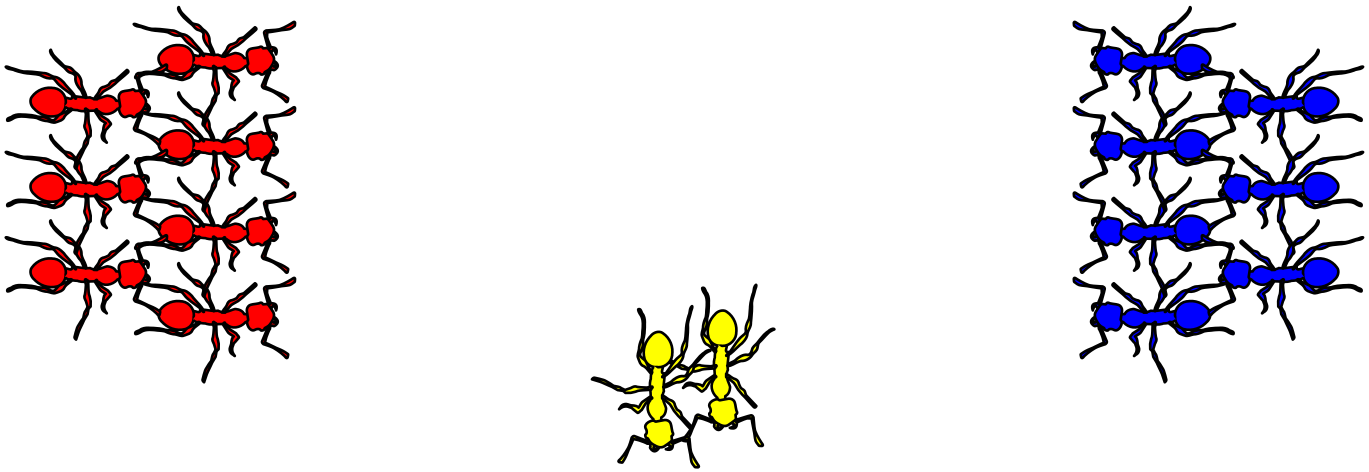
blue vs red:
$40/24 \approx 1.7$

Stage 2: Amplifying majority



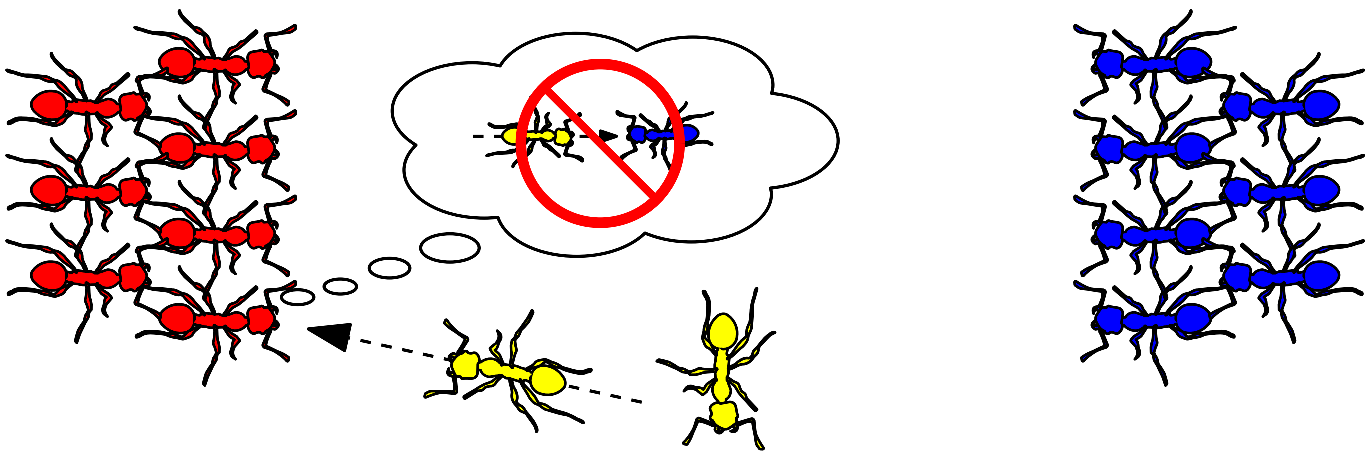
Mathematical Challenges

- Stochastic Dependence



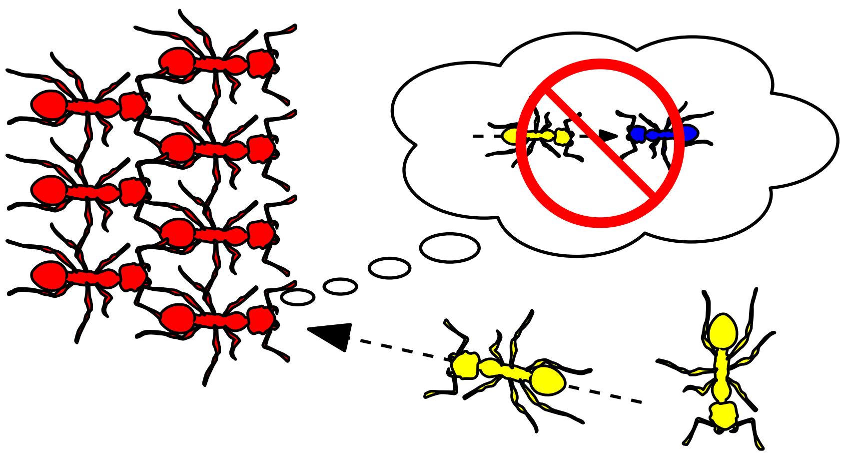
Mathematical Challenges

- Stochastic Dependence

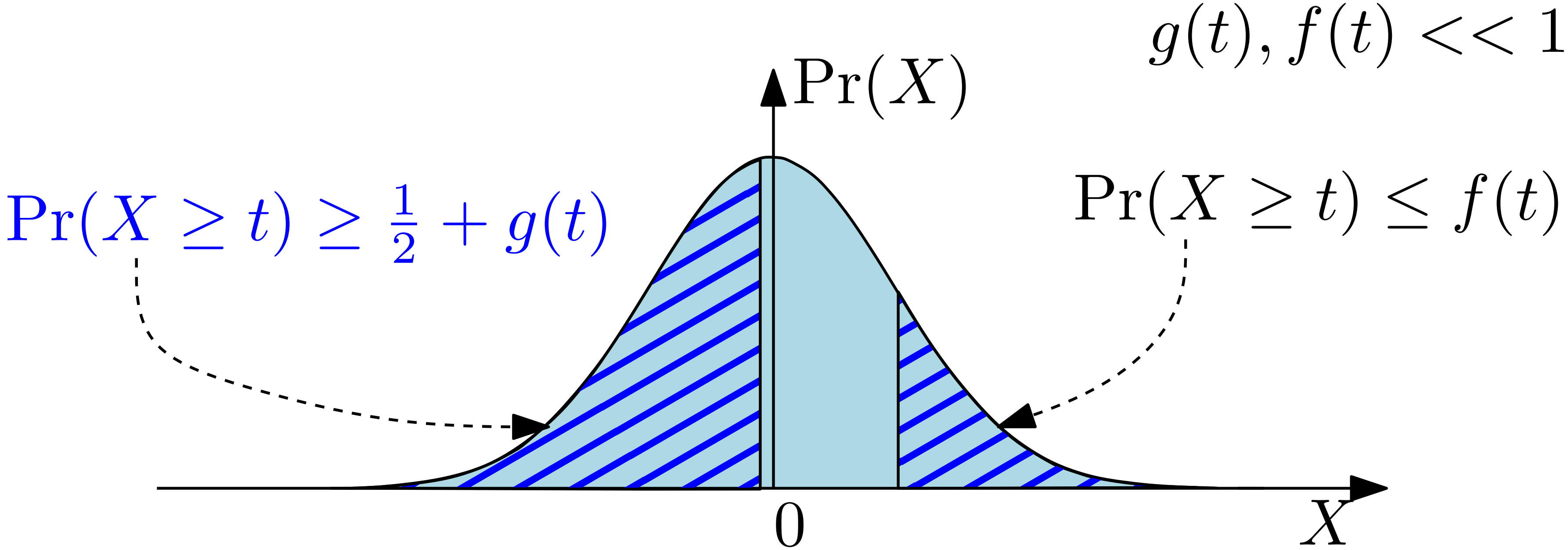


Mathematical Challenges

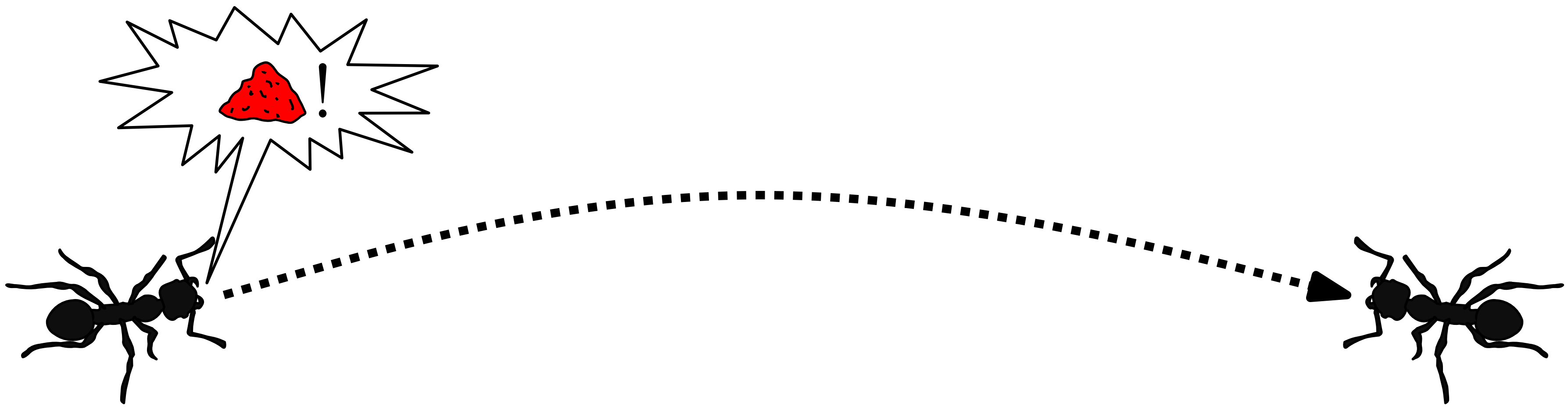
- Stochastic Dependence



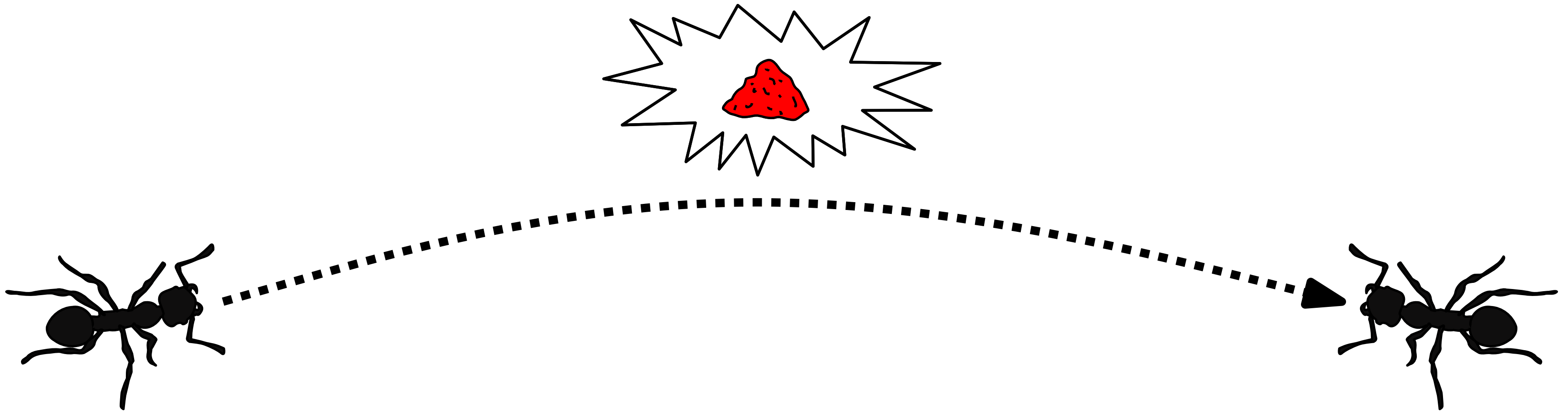
- "Small Deviations"



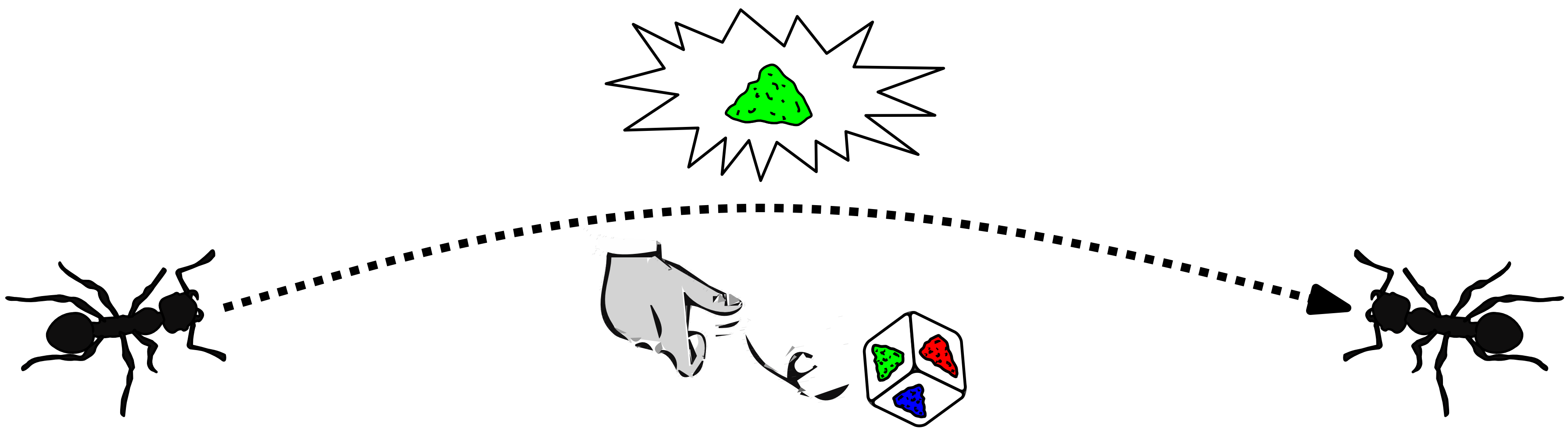
Multivalued Case



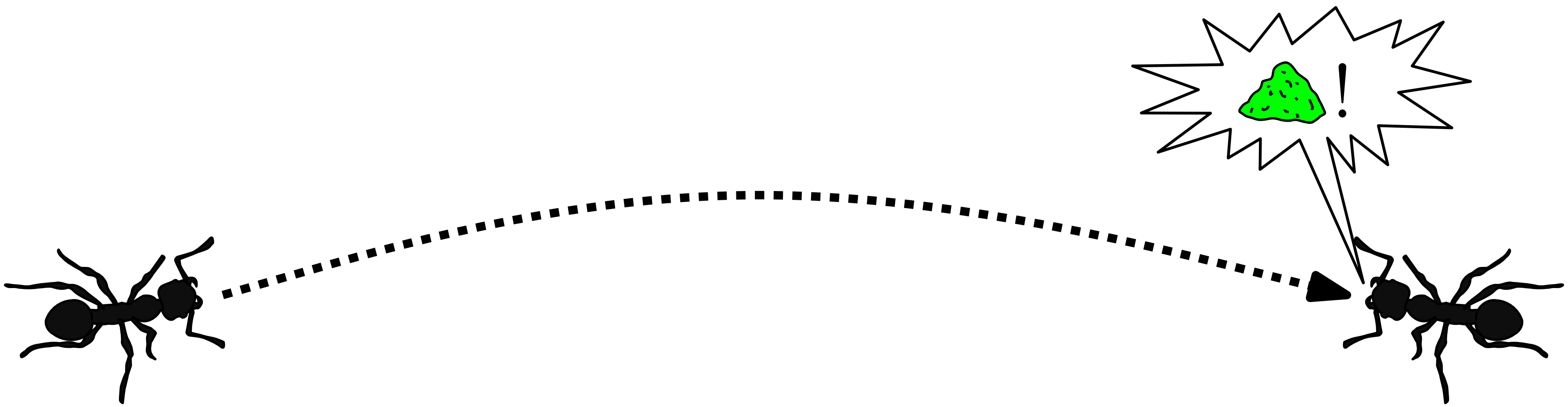
Multivalued Case



Multivalued Case

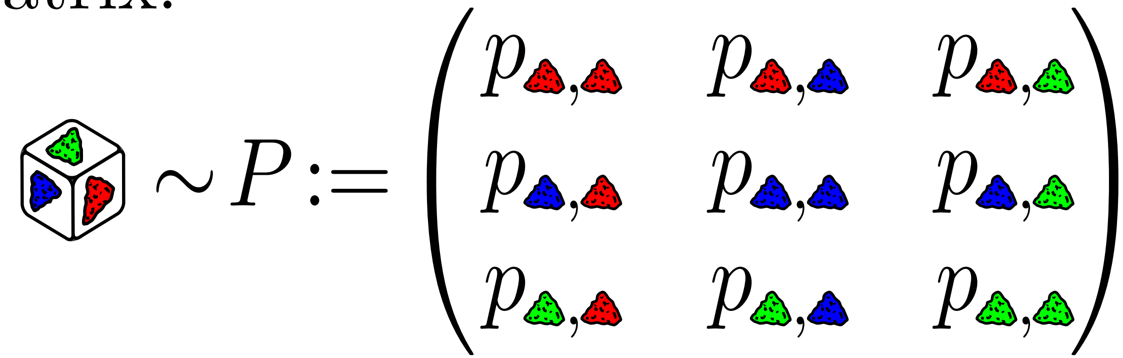


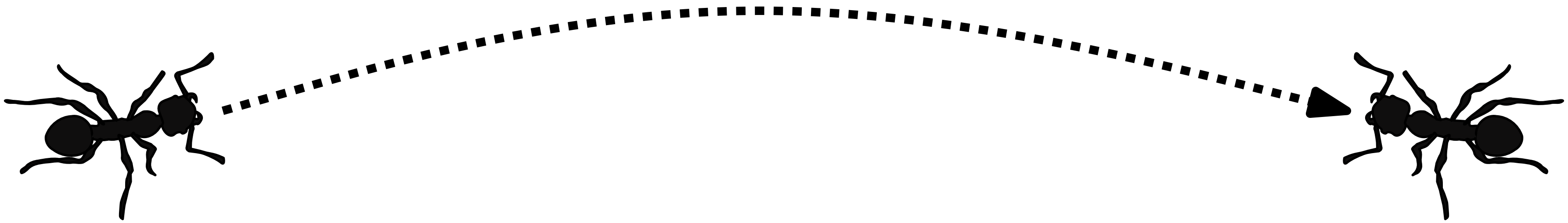
Multivalued Case



Multivalued Case

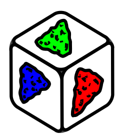
Noise Matrix:



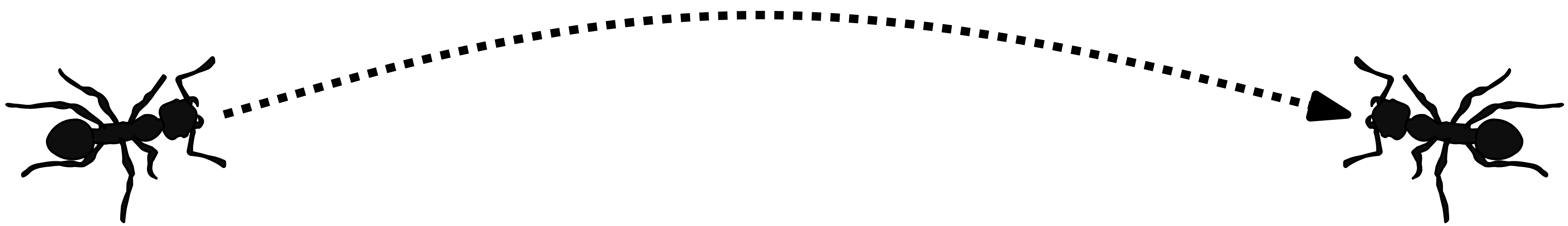


Multivalued Case

Noise Matrix:



$$\sim P := \begin{pmatrix} p_{\text{red},\text{red}} & p_{\text{red},\text{blue}} & p_{\text{red},\text{green}} \\ p_{\text{blue},\text{red}} & p_{\text{blue},\text{blue}} & p_{\text{blue},\text{green}} \\ p_{\text{green},\text{red}} & p_{\text{green},\text{blue}} & p_{\text{green},\text{green}} \end{pmatrix}$$



Configuration $\mathbf{c} := (\#\text{blue}/n, \#\text{red}/n, \#\text{green}/n)$

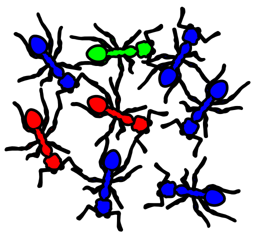
δ-majority-biased configuration w.r.t. blue:

$$\#\text{blue}/n - \#\text{red}/n > \delta$$

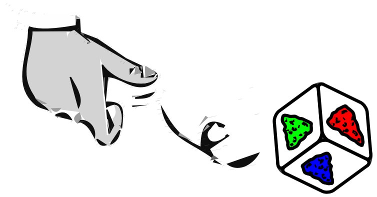
$$\#\text{blue}/n - \#\text{green}/n > \delta$$

Majority-Preserving Matrix

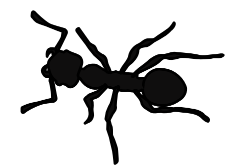
Random sender in conf. $\mathbf{c}$



Noise acting according to matrix P



Message distributed as $c \cdot P$

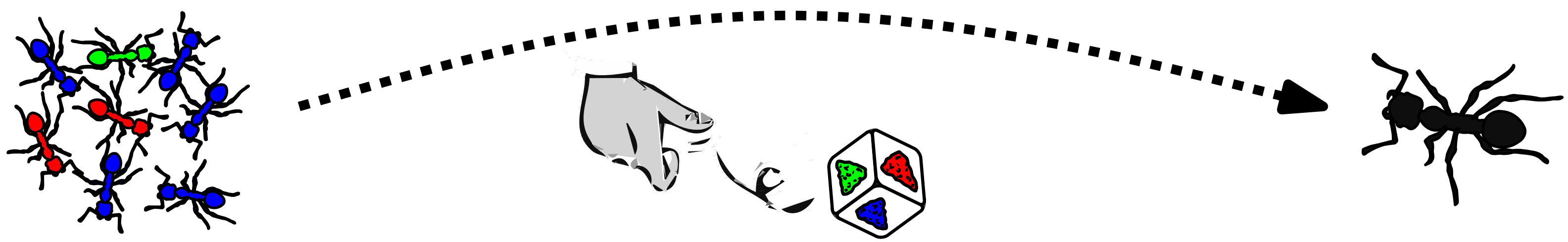


Majority-Preserving Matrix

Random sender in conf. $\mathbf{c}$

Noise acting according to matrix P

Message distributed as $c \cdot P$



(ε, δ)-majority-preserving noise matrix:

$$(\mathbf{c}P)_{\color{blue}\blacktriangle} - (\mathbf{c}P)_{\color{red}\blacktriangle} > \varepsilon\delta$$

$$(\mathbf{c}P)_{\color{blue}\blacktriangle} - (\mathbf{c}P)_{\color{green}\blacktriangle} > \varepsilon\delta$$

Main Result

Theorem. Let S be the initial set of agents with opinions in $[k]$. Suppose that S is $\delta = \Omega(\sqrt{\log n / |S|})$-majority-biased with $|S| = \Omega(\frac{\log n}{\epsilon^2})$ and the noise matrix P is (ϵ, δ)-majority-preserving. Then the plurality consensus problem can be solved in $O(\frac{\log n}{\epsilon^2})$ rounds w.h.p., with $O(\log \log n + \log \frac{1}{\epsilon})$ memory per node.

Main Result

Theorem. Let S be the initial set of agents with opinions in $[k]$. Suppose that S is $\delta = \Omega(\sqrt{\log n/|S|})$-majority-biased with $|S| = \Omega(\frac{\log n}{\epsilon^2})$ and the noise matrix P is (ϵ, δ)-majority-preserving. Then the plurality consensus problem can be solved in $O(\frac{\log n}{\epsilon^2})$ rounds w.h.p., with $O(\log\log n + \log\frac{1}{\epsilon})$ memory per node.

$|S| = 1 \implies$ rumor spreading in $O(\frac{\log n}{\epsilon^2})$ rounds

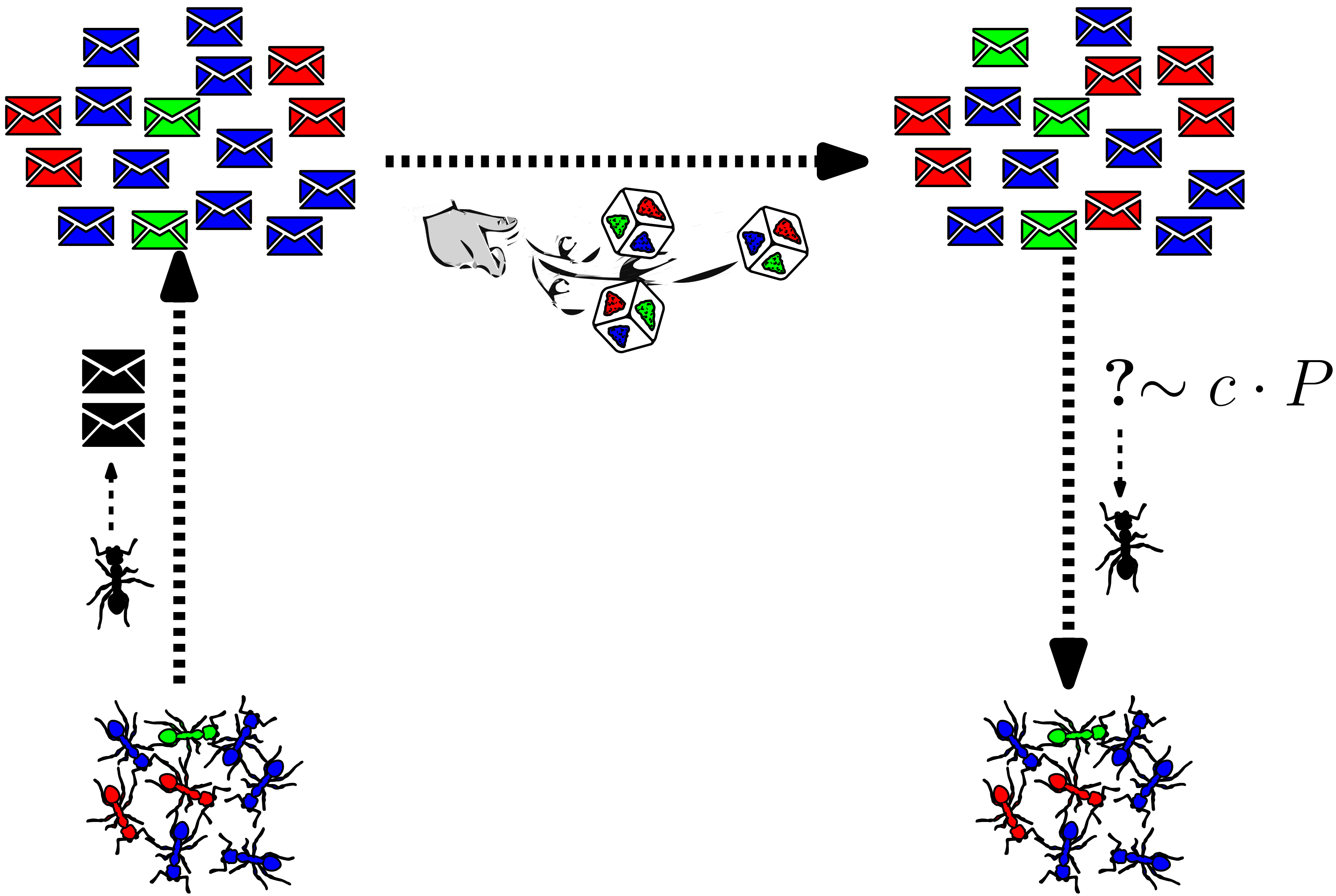
Main Result

Theorem. Let S be the initial set of agents with opinions in $[k]$. Suppose that S is $\delta = \Omega(\sqrt{\log n/|S|})$-majority-biased with $|S| = \Omega(\frac{\log n}{\epsilon^2})$ and the noise matrix P is (ϵ, δ)-majority-preserving. Then the plurality consensus problem can be solved in $O(\frac{\log n}{\epsilon^2})$ rounds w.h.p., with $O(\log \log n + \log \frac{1}{\epsilon})$ memory per node.

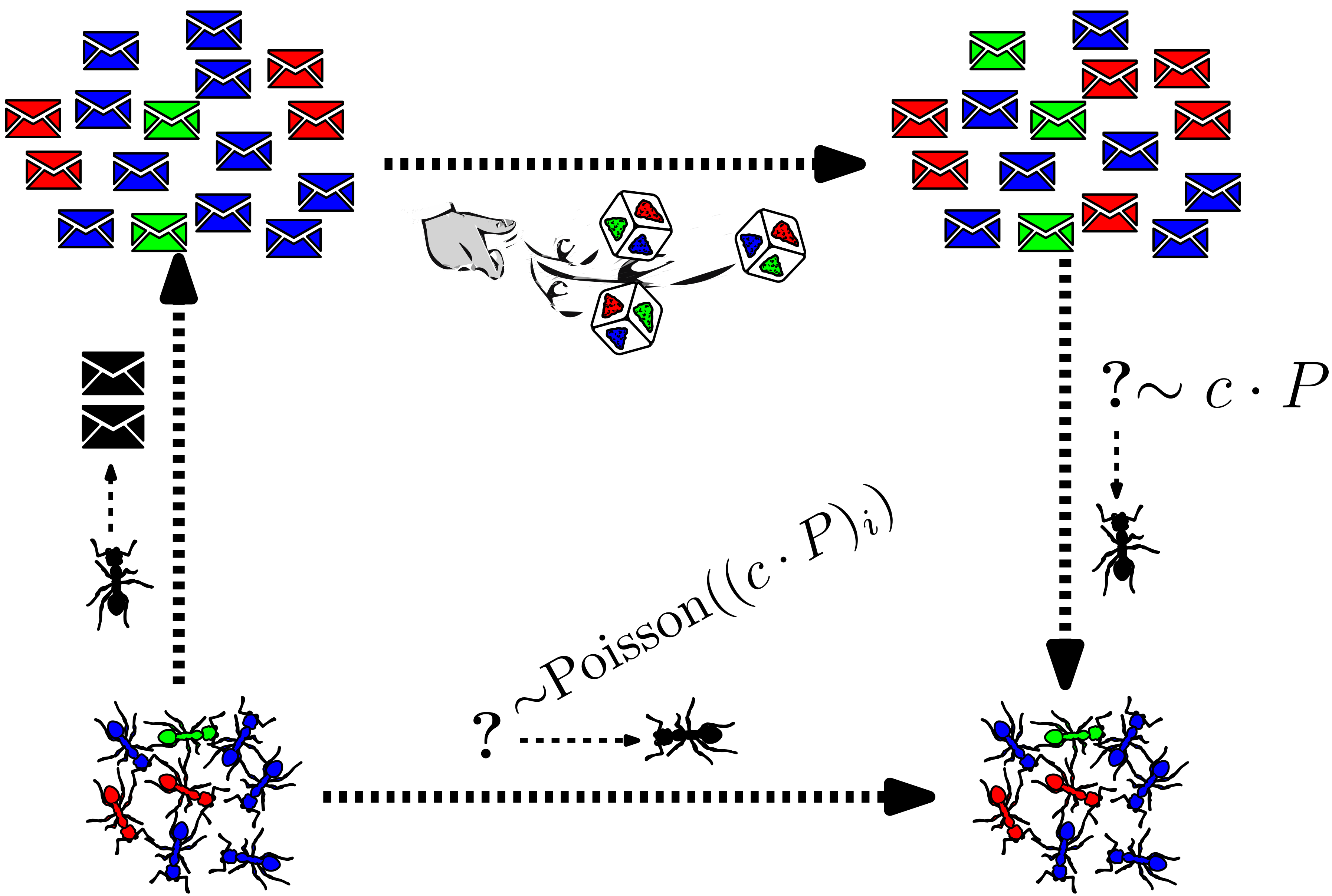
$|S| = 1 \implies$ rumor spreading in $O(\frac{\log n}{\epsilon^2})$ rounds

$$P = \begin{pmatrix} 1/2 + \varepsilon & 1/2 - \varepsilon \\ 1/2 - \varepsilon & 1/2 + \varepsilon \end{pmatrix} \implies \text{Feinerman et al.}$$

Poisson Approximation



Poisson Approximation



Poisson Approximation

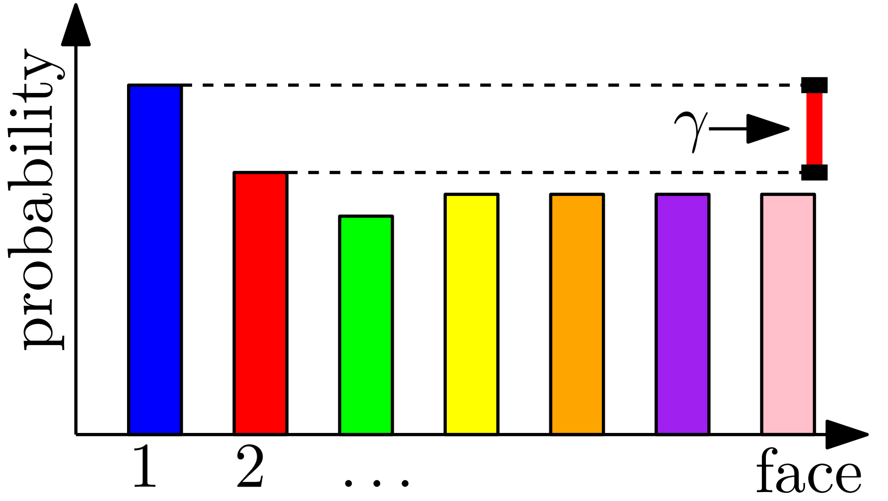
Lemma. balls-in-bins experiment:
- h colored balls are thrown in n bins, h_i balls have color $1 \le i \le k$,
- $\{X_{u,i}\}_{u\in\{1,\dots,n\},i\in\{1,\dots,k\}}$ number of i-colored balls that end up in bin u,
- f non-negative function with $\mathbb{Z}_{\ge 0}$ arguments $\{x_{u,i}\}_{u\in\{1,\dots,n\},i\in\{1,\dots,k\}}$ and z,
- $\{Y_{u,i}\}_{u\in\{1,\dots,n\},i\in\{1,\dots,k\}}$ independent r.v. with $Y_{u,i} \sim$Poisson(h_i/n) and Z integer valued r.v. independent from $X_{u,i}$s and $Y_{u,i}$s.

$$\mathbb{E}\left[f\left(X_{1,1},\dots,X_{n,1},X_{n,2},\dots,X_{n,k},Z\right)\right]$$
$$\le e^k\sqrt{\prod_i h_i}\,\mathbb{E}\left[f\left(Y_{1,1},\dots,Y_{n,1},Y_{n,2},\dots,Y_{n,k},Z\right)\right].$$

Corollary. Given conf. $\mathbf{c}$, if event $\mathcal{E}$ holds in process $\mathbf{P}$ with prob $1-n^{-b}$ with $b > (k\log h)/(2\log n)$, then it holds w.h.p. also in the original process.

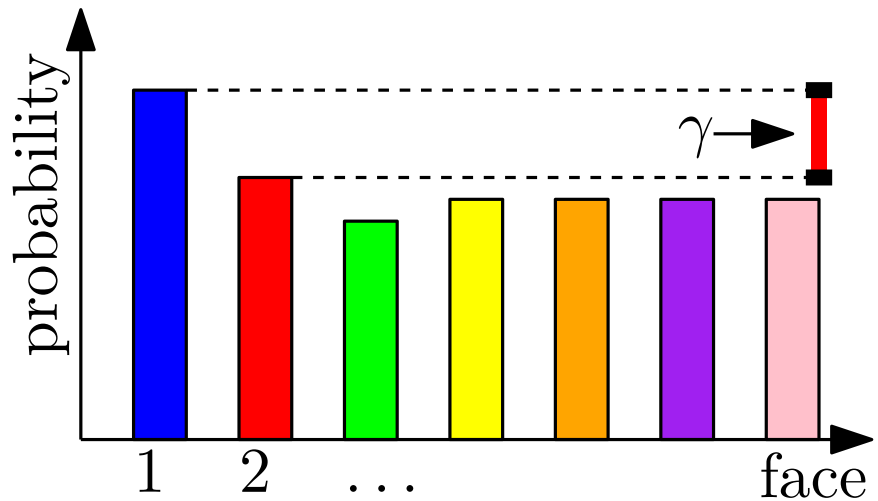
Probability Amplification

A dice with k faces is thrown ℓ times.



Probability Amplification

A dice with k faces is thrown ℓ times.



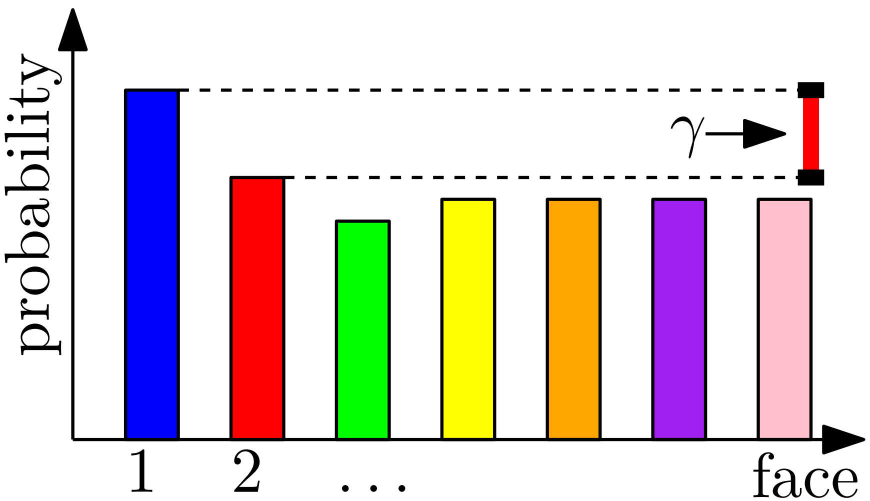
$\mathcal{M}$:= most frequent face in the ℓ throws (breaking ties at random).

For any $j \neq 1$

$$\Pr(\mathcal{M} = 1) - \Pr(\mathcal{M} = j) \geq \text{const} \cdot \sqrt{\ell}\,\gamma(1-\gamma^2)^{\frac{\ell-1}{2}}$$

Probability Amplification

A dice with k faces is thrown ℓ times.



$\mathcal{M}$:= most frequent face in the ℓ throws (breaking ties at random).

For any $j \neq 1$

$$\Pr(\mathcal{M} = 1) - \Pr(\mathcal{M} = j) \geq \text{const} \cdot \sqrt{\ell}\,\gamma(1-\gamma^2)^{\frac{\ell-1}{2}}$$

open problem: const $\approx e^{-\Theta(k)}$

Binomial vs Beta

Given $p \in (0,1)$ and $0 \le j \le \ell$ it holds

$$\Pr\left(Bin(n,p) \le j\right) = \sum_{j<i\le\ell} \binom{\ell}{i} p^{i} \left(1-p\right)^{\ell-i}$$

$$= \binom{\ell}{j+1} (j+1) \int_0^p z^{j} \left(1-z\right)^{\ell-j-1} dz$$

$$= \Pr\left(Beta(n-k,k+1) < 1-p\right).$$

Binomial vs Beta

Given $p \in (0,1)$ and $0 \leq j \leq \ell$ it holds

$$\Pr\left(Bin(n,p) \leq j\right) = \sum_{j<i\leq \ell} \binom{\ell}{i} p^i \left(1-p\right)^{\ell-i}$$

$$= \binom{\ell}{j+1} (j+1) \int_0^p z^j \left(1-z\right)^{\ell-j-1} dz$$

$$= \Pr\left(Beta(n-k,k+1) < 1-p\right).$$

Multinomial vs Dirichlet?

Thank You!