Find Your Place: Simple Distributed Algorithms for Community Detection[‡]

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joint work with

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[‡]preprint at **goo.gl/aqZmCD**

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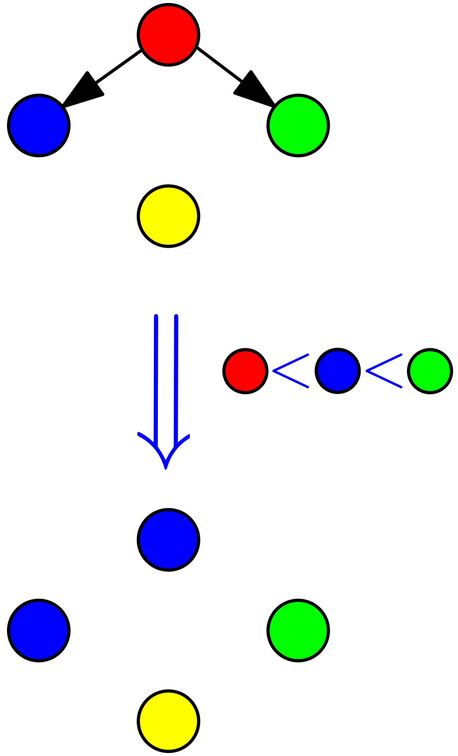

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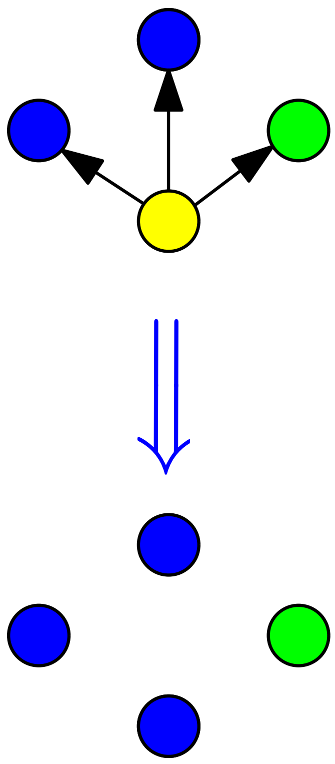

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- Undecided-state dynamics
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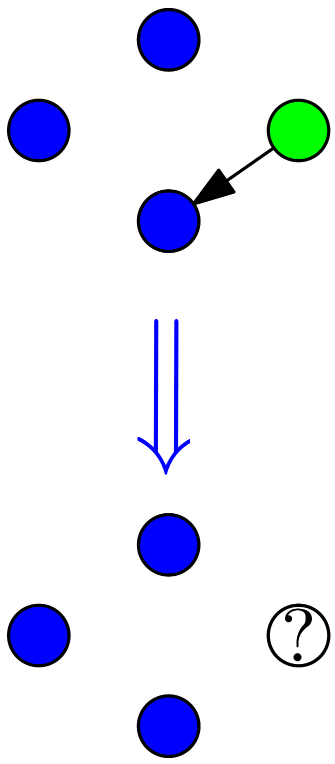

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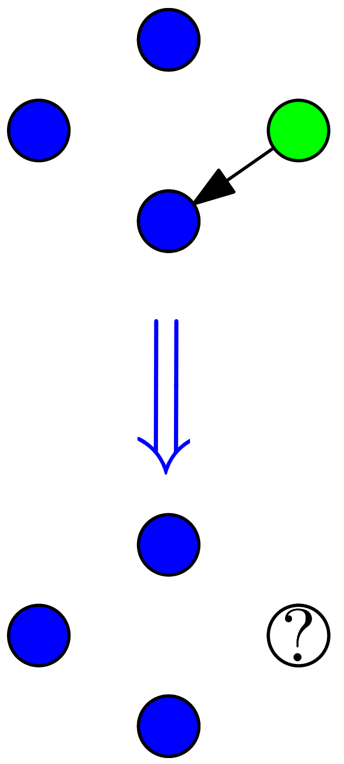


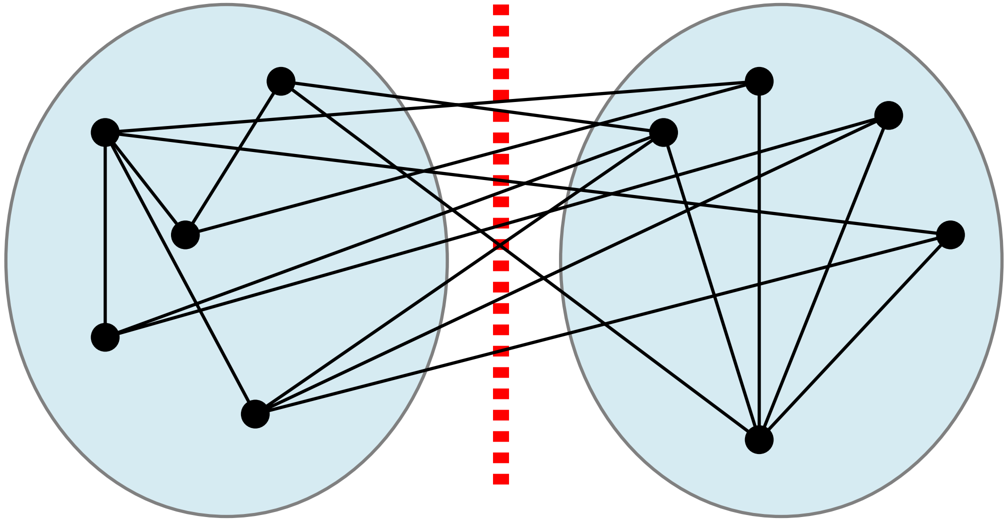
Can dynamics solve a problem non-trivial in centralized setting?

Community Detection as Minimum Bisection

Minimum Bisection Problem.

Input: a graph G with $2n$ nodes.

Output: $S = \arg\min_{\substack{S \subset V \\ |S|=n}} E(S, V-S)$.

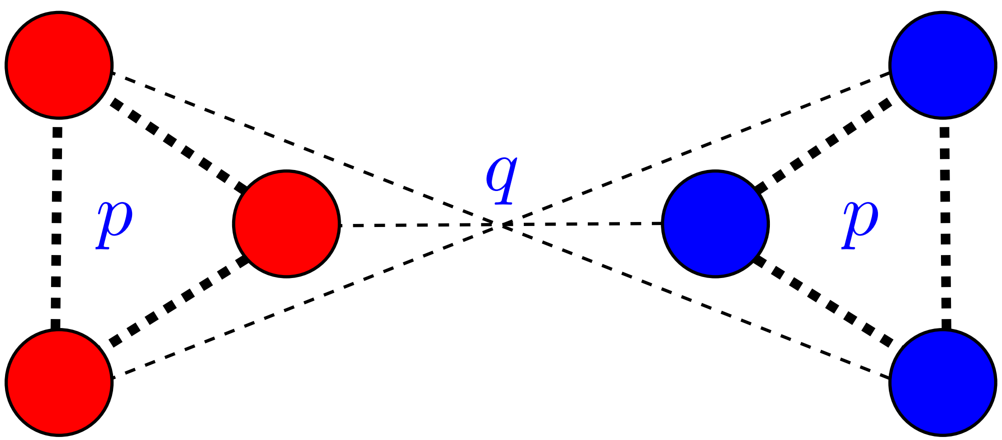


[Garey, Johnson, Stockmeyer '76]:

Min-Bisection is *NP-Complete*.

The Stochastic Block Model

Stochastic Block Model (SBM). Two "communities" of equal size V_1 and V_2, each edge inside a community included with probability $p = \frac{a}{n}$, each edge across communities included with probability $q = \frac{b}{n} < p$.



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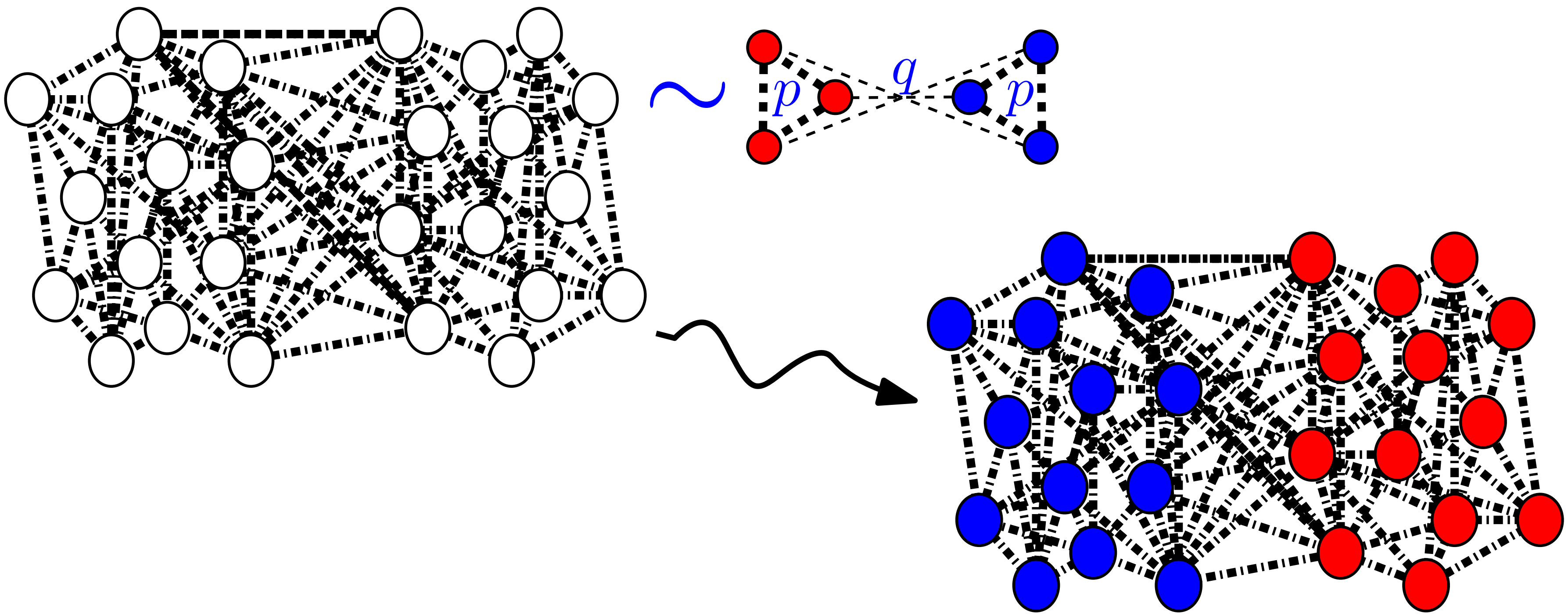
Reconstruction problem. Given graph generated by SBM, find original partition.



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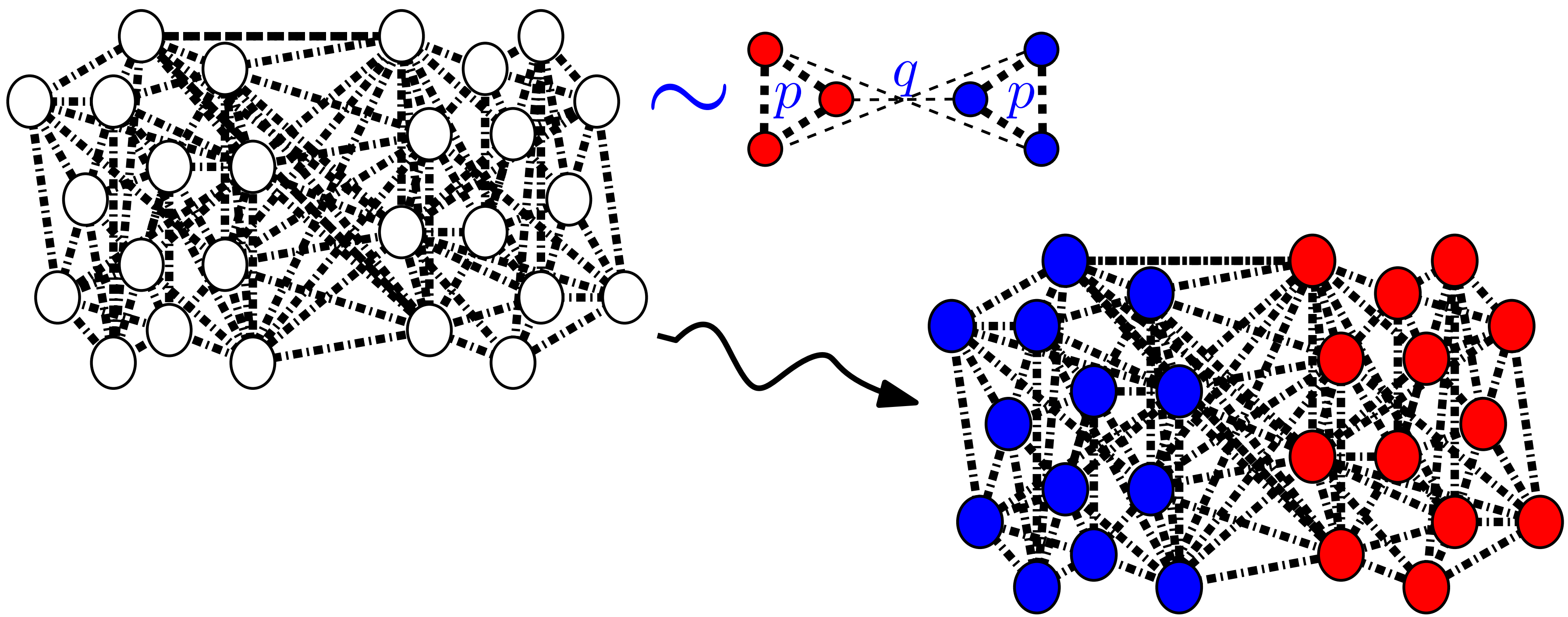
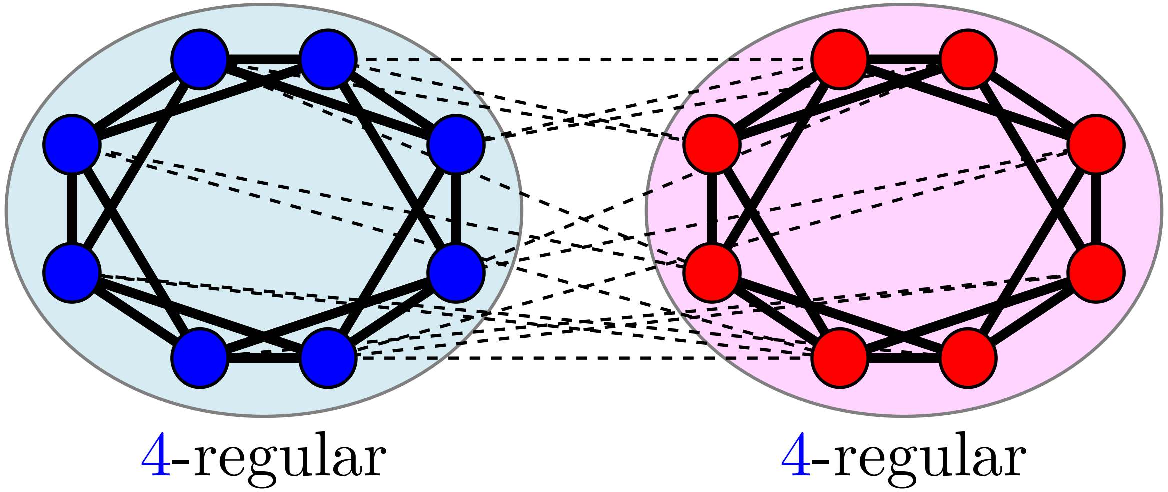


$\lambda_2(P) \approx \frac{a-b}{d} \implies$ mixing time of a random walk on $\mathcal{G}_{2n,\frac{a}{n},\frac{b}{n}}$ is $\geq \frac{1}{1-\lambda_2} \approx \frac{a+b}{2b}$.

Regular Stochastic Block Model

Regular SBM (RSBM) [Brito et al. SODA'16]. A graph $G = (V_1 \dot{\bigcup} V_2, E)$ s.t.

- $|V_1| = |V_2|$,
- $G\big|_{V_1}, G\big|_{V_2}$ $\sim$ random a-regular graphs
- $G\big|_{E(V_1,V_2)}$ $\sim$ random b-regular bipartite graph.



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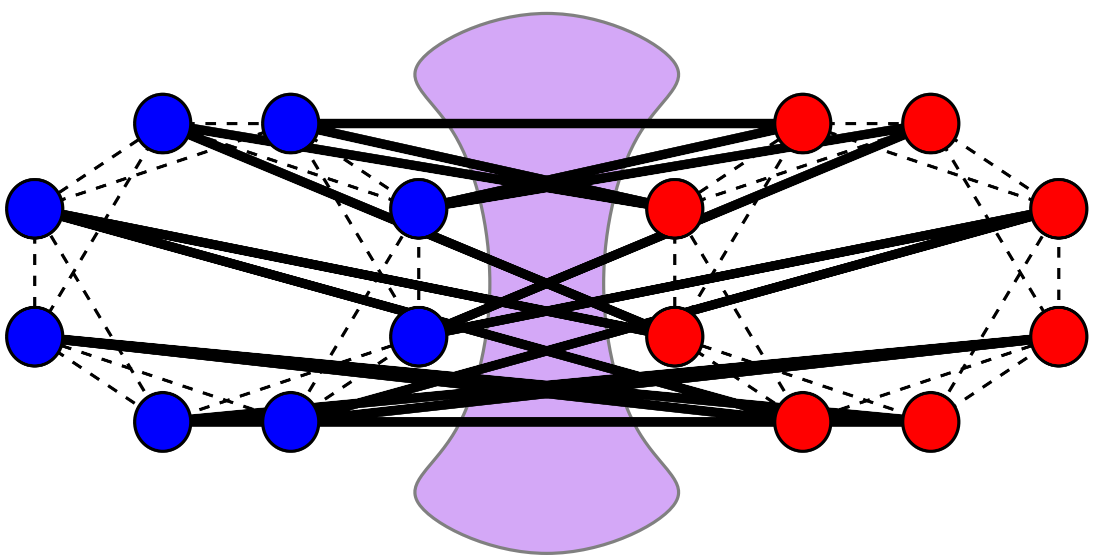

2-regular bipartite

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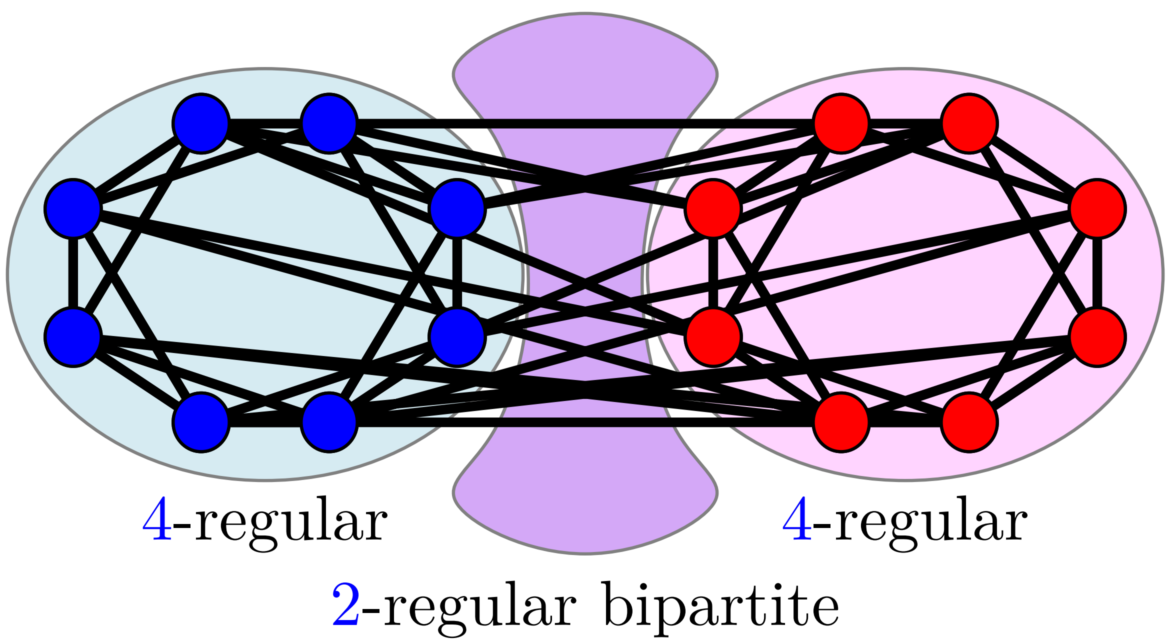

When is Reconstruction Possible?

[Decelle, Massoulie, Mossel, Brito, Abbe et al.]:
Reconstruction is possible iff

- $a - b > 2\sqrt{d}$ in SBM (weak)
- $a - b > 2(\sqrt{a} - \sqrt{b})\sqrt{b} + 2\log n$ in SBM (strong)
- $a - b > 2\sqrt{d-1}$ in RSBM (strong)

Linearizations of *Belief Propagation*, advanced spectral methods (power and Lanczos method), SDP.

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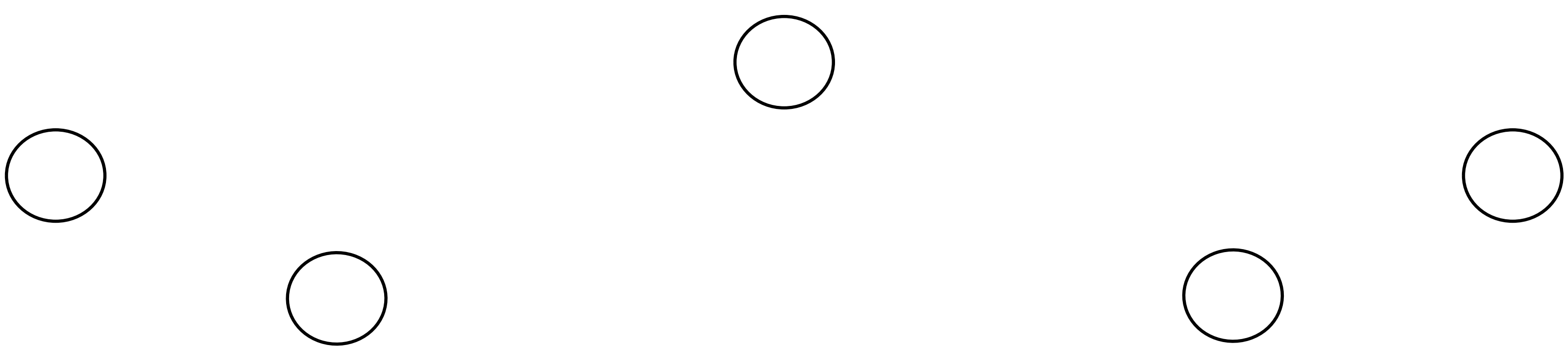
Not a dynamics: nonlinear, different messages to different neighbors

Centralized, not easy to make distribute

The Average Dynamics

Al nodes at the same time:

- At $t = 0$, randomly pick value $x^{(t)} \in \{\mathbf{+1}, \mathbf{-1}\}$.
- Then, at each round
  1. Set value $x^{(t)}$ to average of neighbors,
  2. Set label to **blue** if $x^{(t)} < x^{(t-1)}$, **red** otherwise.



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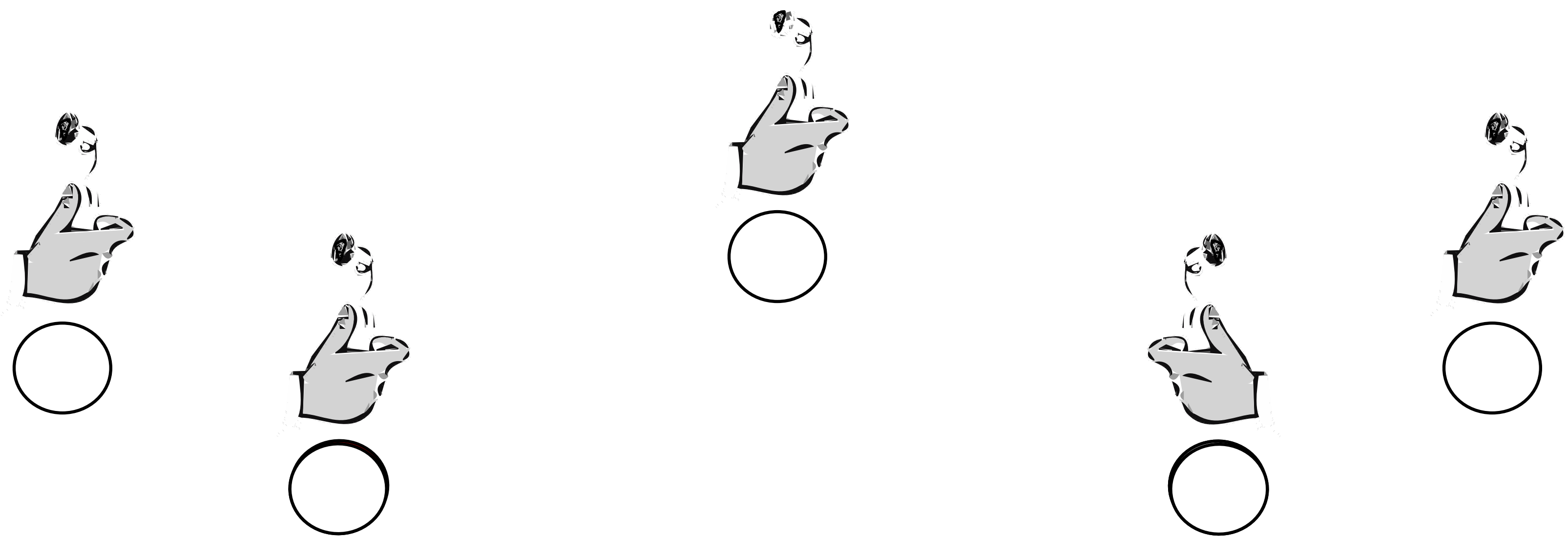

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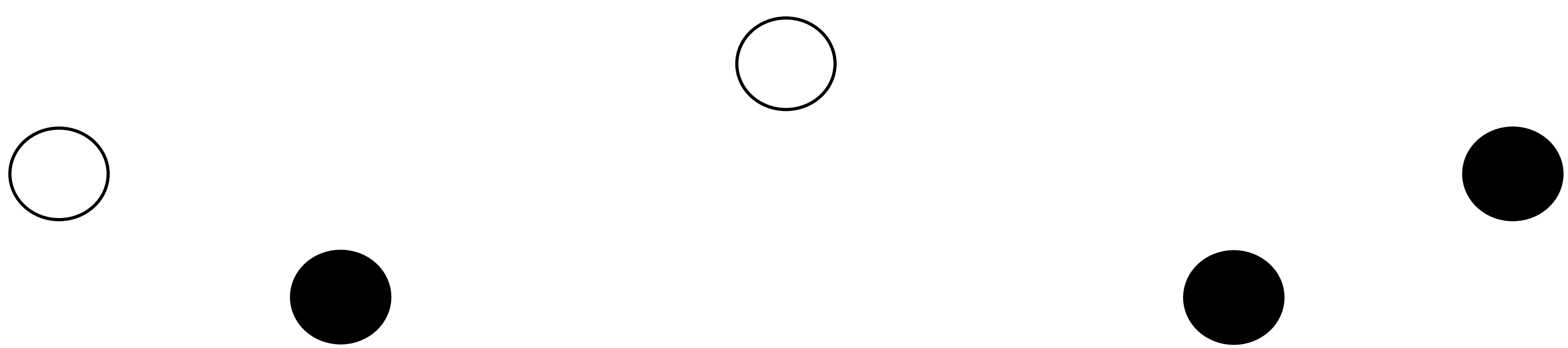

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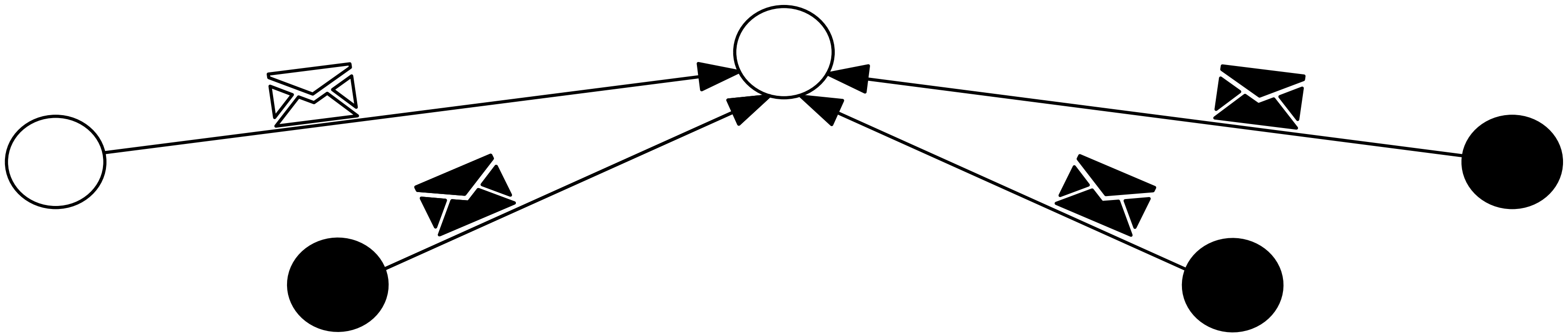

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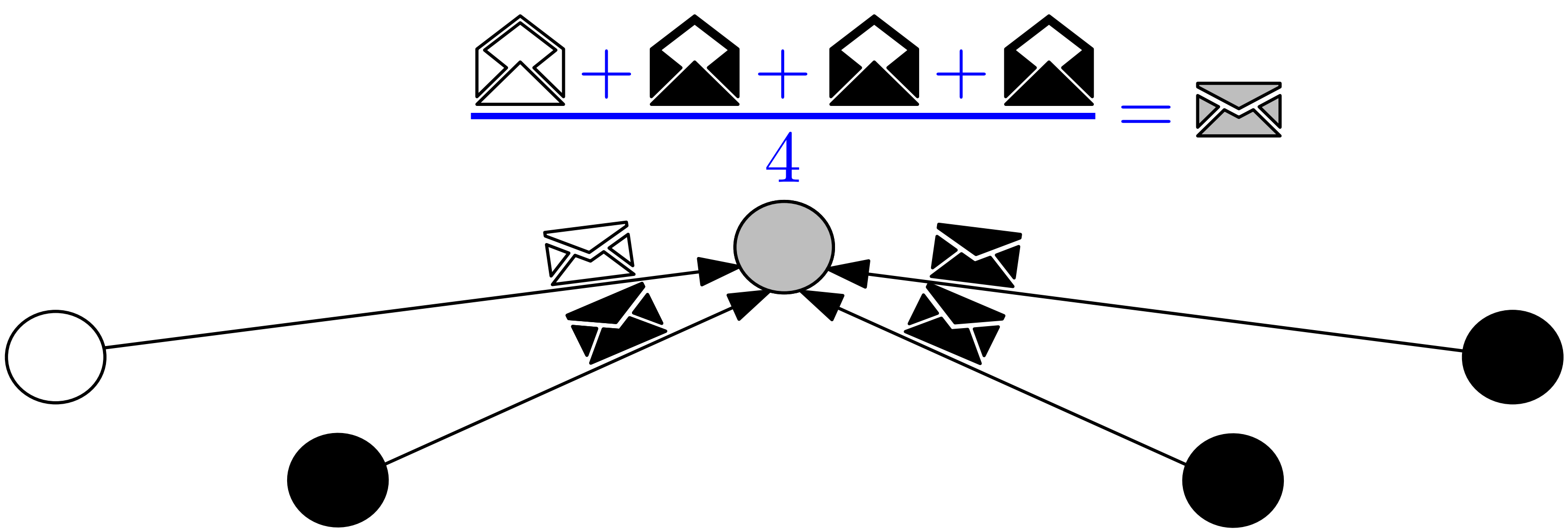

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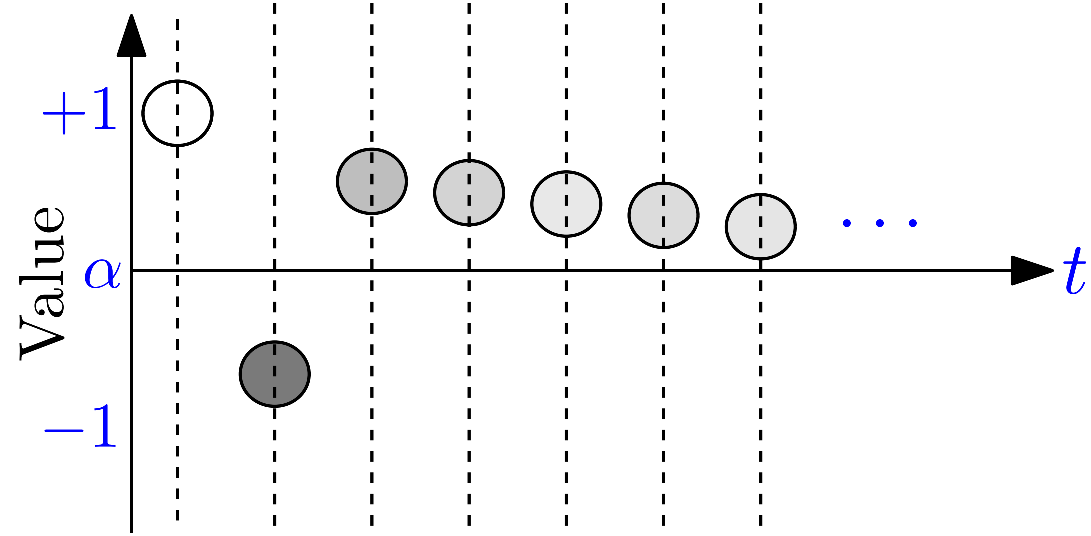

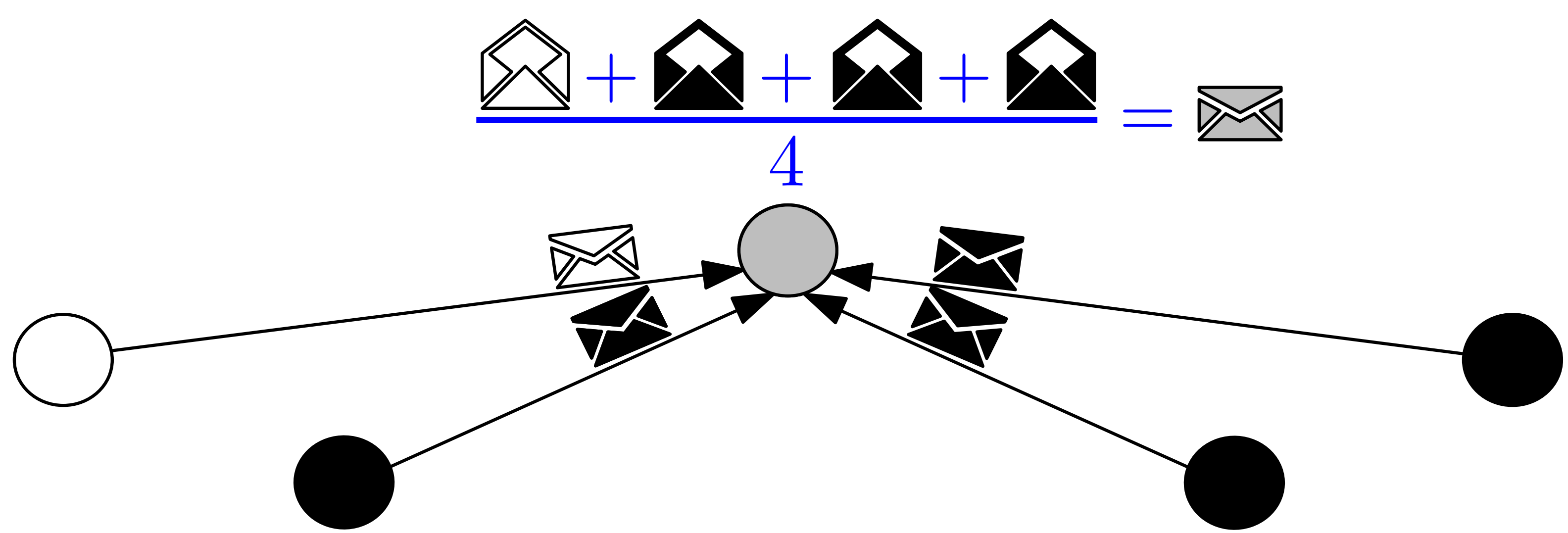

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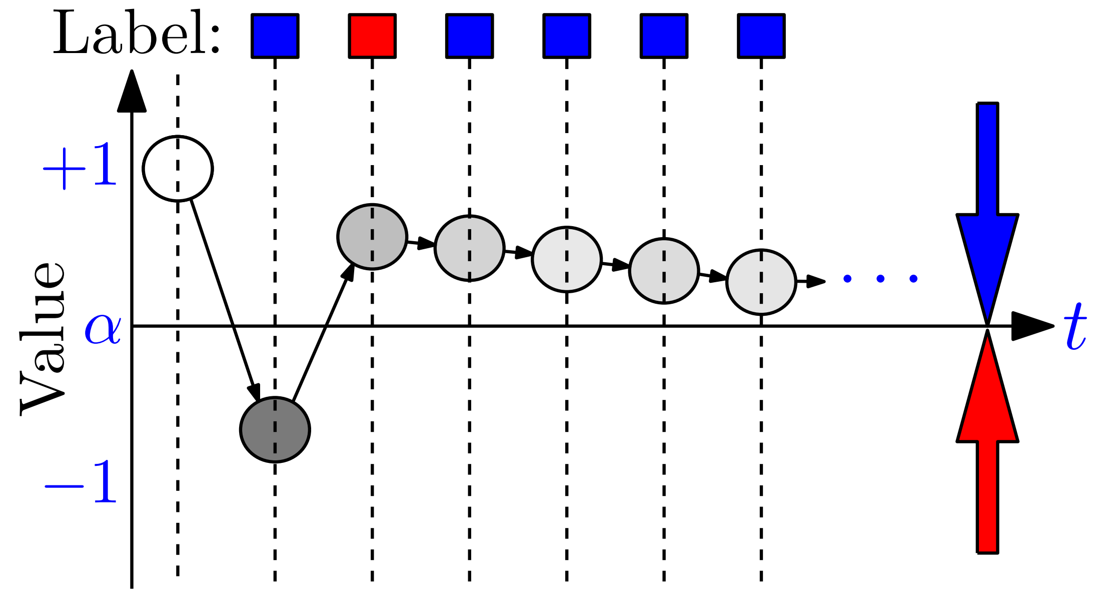

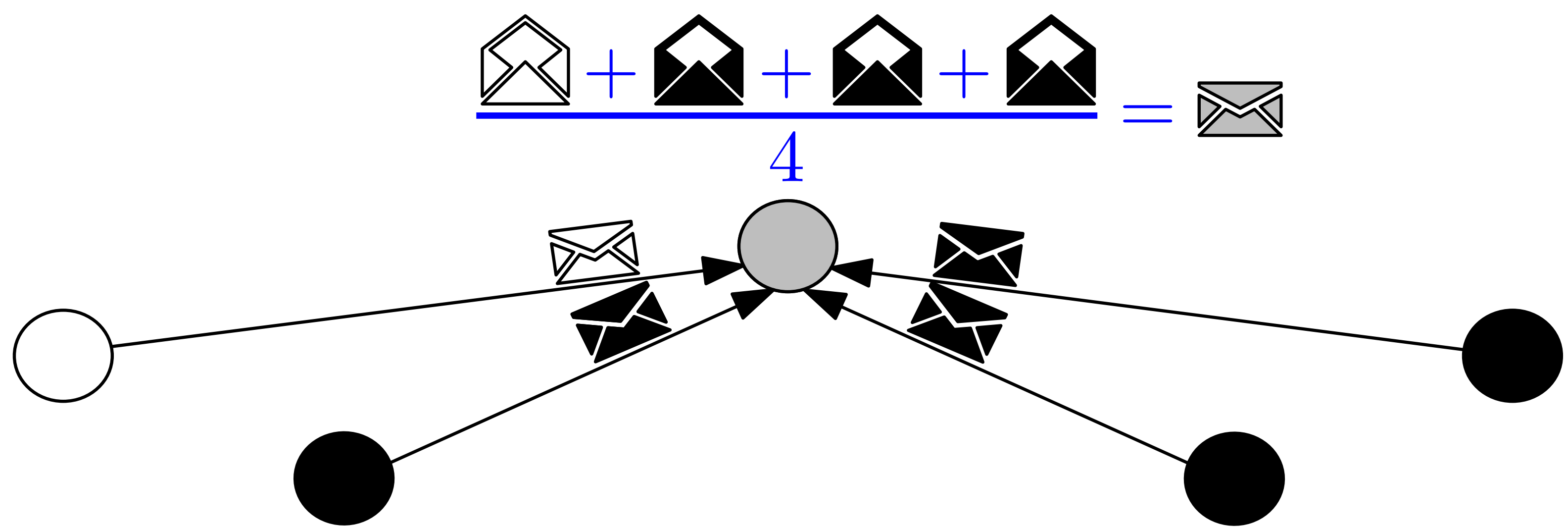

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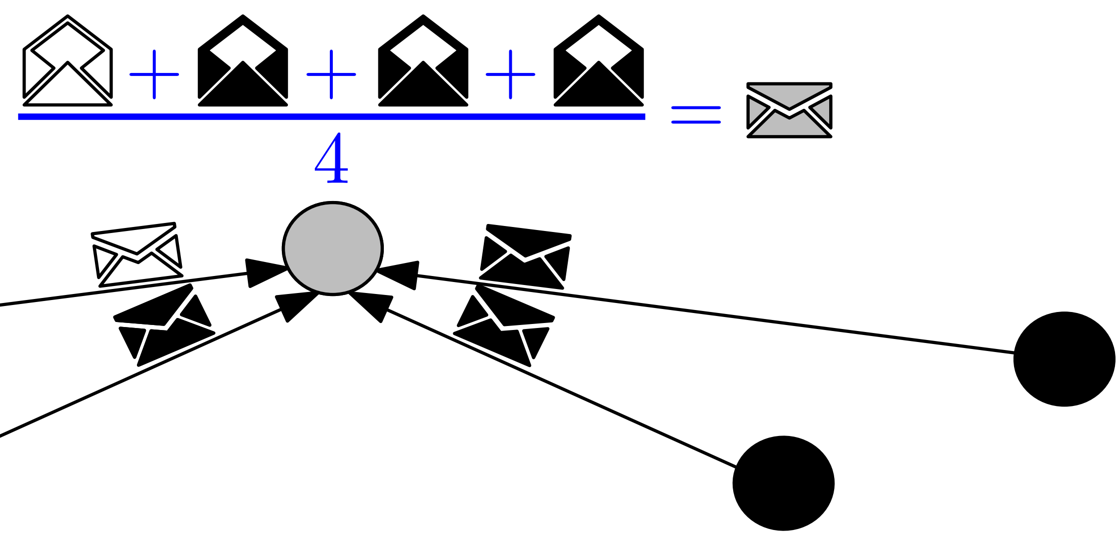
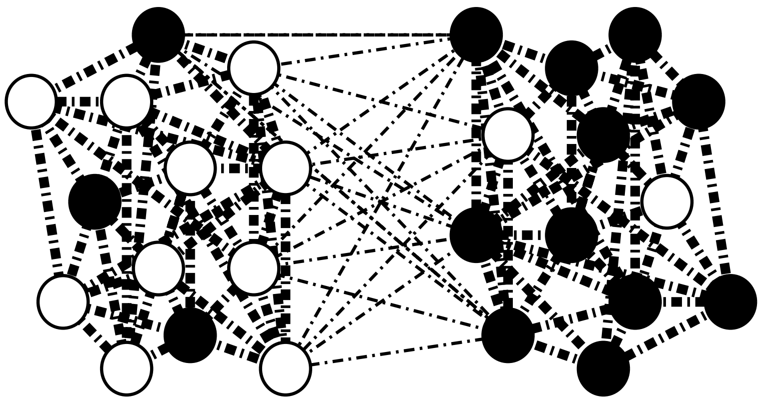
Well studied process [Shah '09]:

- Converges to (weighted) global average of initial values,
- Convergence time $=$ mixing time of G,
- Important applications in fault-tolerant self-stabilizing consensus.



Our Results



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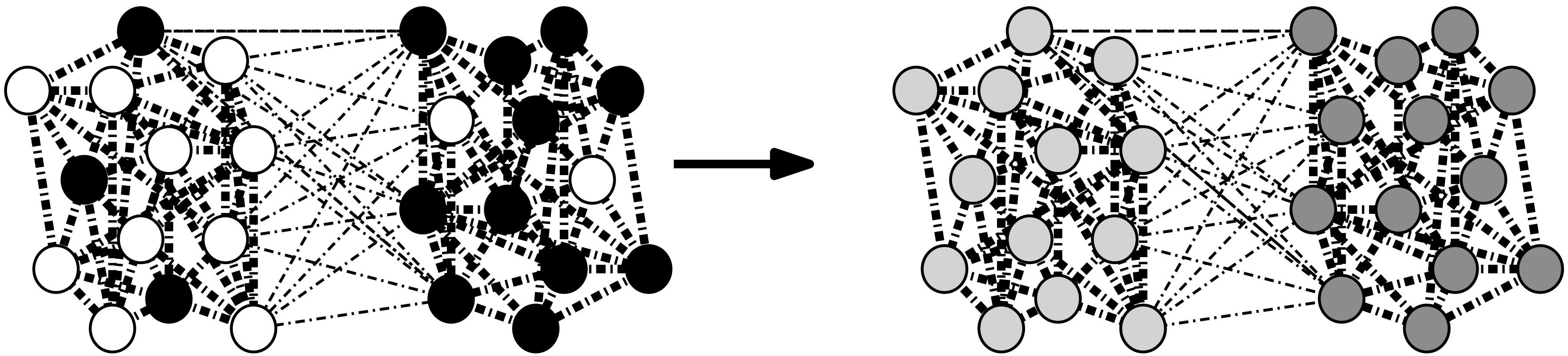

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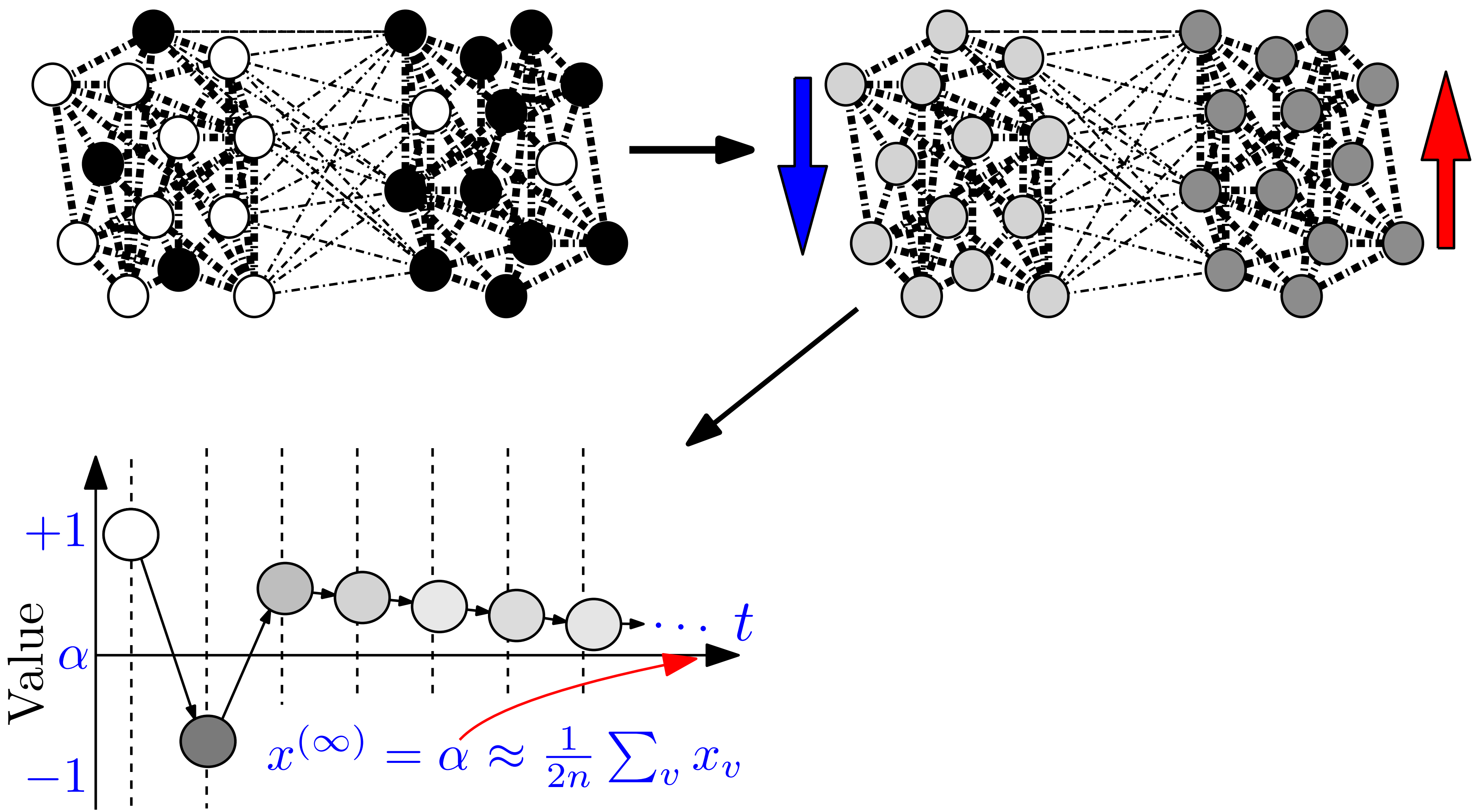

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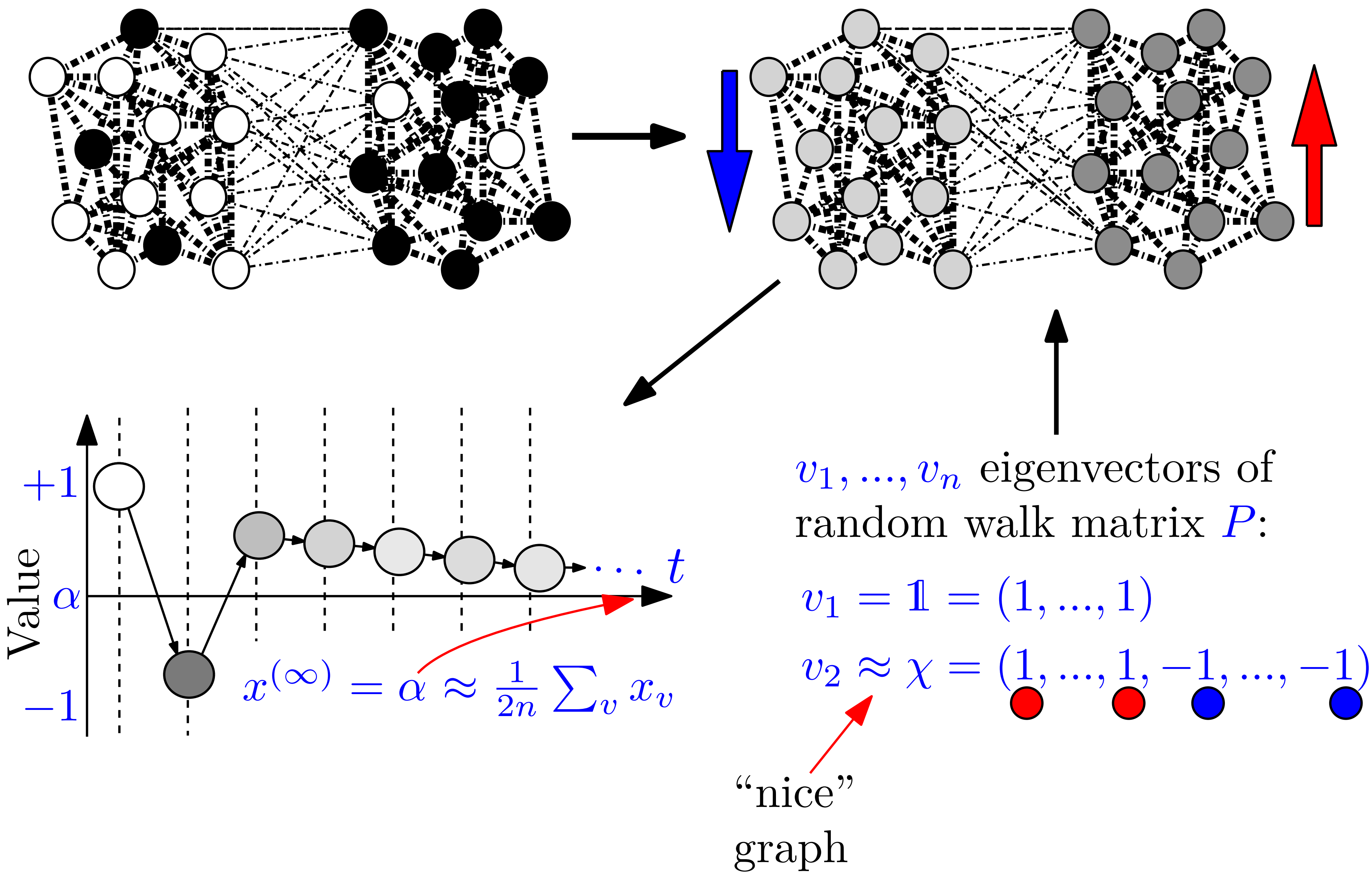

$v_1, \ldots, v_n$ eigenvectors of
random walk matrix P:
$v_1 = \mathbb{1} = (1, \ldots, 1)$
$v_2 \approx \chi = (1, \ldots, 1, -1, \ldots, -1)$
"nice"
graph
Value
$+1$
α
-1
$\cdots\ t$
$x^{(\infty)} = \alpha \approx \frac{1}{2n}\sum_v x_v$

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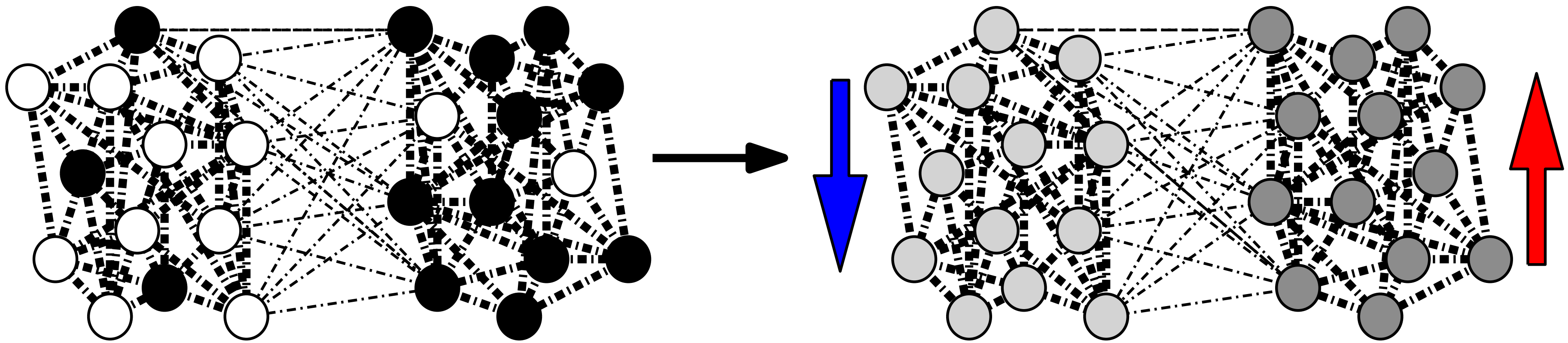

(Informal) Theorem. $G = (V_1 \dot{\bigcup} V_2, E)$ s.t.
i) $\chi = \mathbf{1}_{V_1} - \mathbf{1}_{V_2}$ close to right-eigenvector of eigenvalue λ_2 of transition matrix of G, and
ii) gap between λ_2 and $\lambda = \max\{\lambda_3, |\lambda_n|\}$ sufficiently large, then
Averaging (approximately) identifies (V_1, V_2).

Properties of the Averaging Dynamics

Al nodes at the same time:

- At $t = 0$, randomly pick value $x^{(t)} \in \{\textbf{blue}, \textbf{red}\}$.
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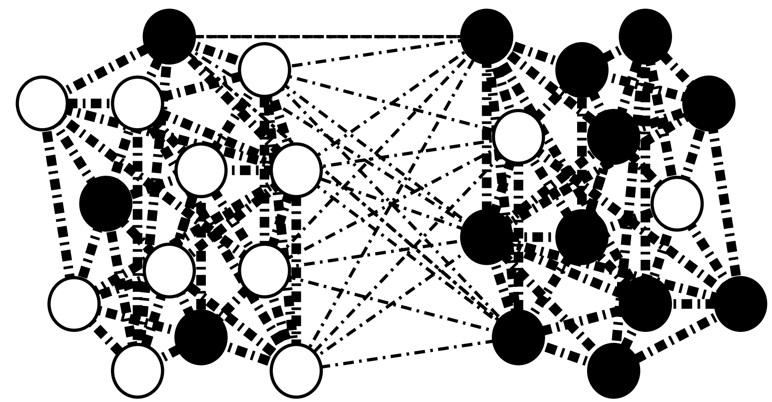
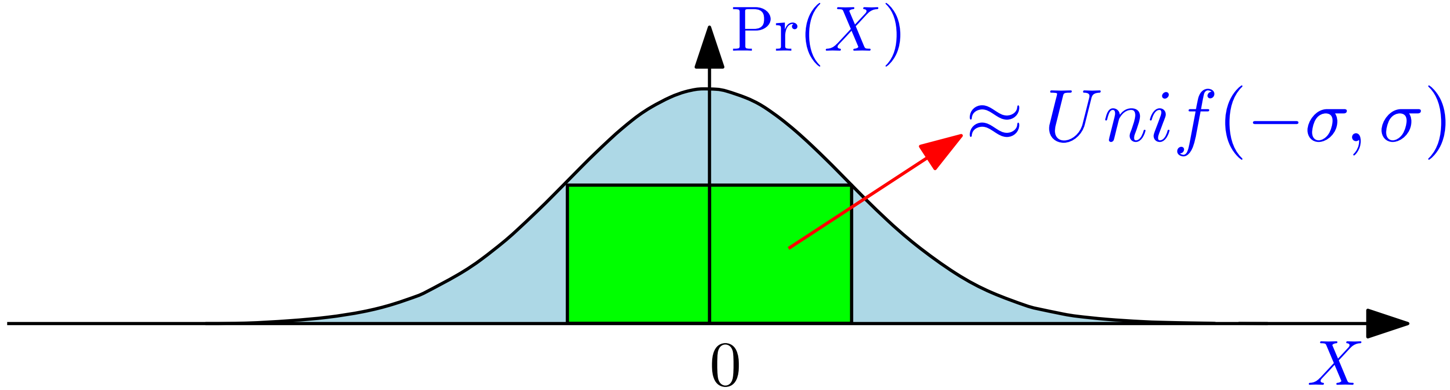


$$\Pr\left(\left|\sum_{v\in V_1}\mathbf{x}(v) - \sum_{v\in V_2}\mathbf{x}(v)\right| > n^{\epsilon}\right) \geq 1 - n^{\Omega(1)} \text{ (w.h.p.)}$$



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Features:

- No explicit eigenvector computation
- Implicit "simulation" of power method

$A = (\mathbb{1}_{((u,v)\in E)})_{u,v\in V}$ adjacency matrix of G

D diagonal matrix of node degrees in G

$P = D^{-1}A$ transition matrix of random walk

Averaging is a **linear** dynamics

$$\mathbf{x}^{(t)} = \begin{pmatrix} \bigcirc \\ \bullet \\ \bigcirc \\ \bullet \\ \bullet \end{pmatrix}$$

$$\boxed{\mathbf{x}^{(t)} = P \cdot \mathbf{x}^{(t-1)} = P^t \cdot \mathbf{x}^{(0)}}$$

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Remove projection on first eigenspace $\Longrightarrow$ running time depending on λ_2/λ

Bottleneck of mixing time for spectral methods:

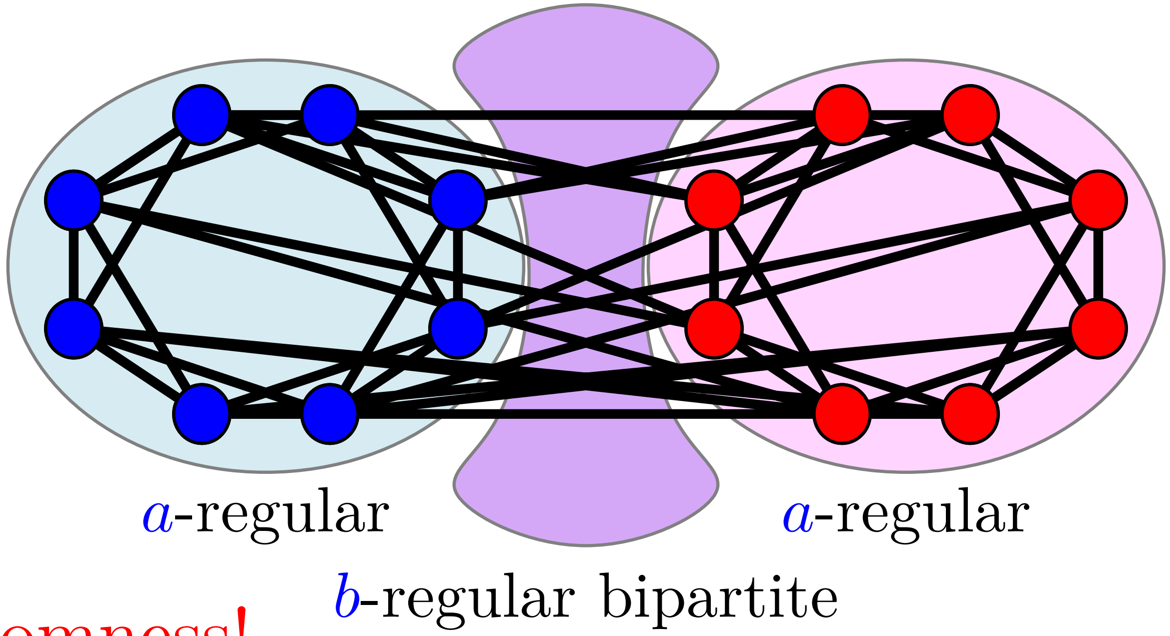
Distributed computation of second eigenvector [Kempe & McSherry '08]: $\mathcal{O}(\tau_{mix} \log^2 n)$.

Regular Clustered and Clustered Graphs

$(2n, d, b)$-clustered Regular Graph.

A graph $G = (V_1 \dot\bigcup V_2, E)$ s.t.

- $|V_1| = |V_2|$,
- G is d regular,
- each $v \in V_i$ has b neighbors in V_{3-i}.



No randomness!

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Thm. If $G\big|_{V_1}$, $G\big|_{V_2}$ expanders and $\lambda_2/\lambda > 1$ (e.g. if $b \ll d/2$), averaging produces strong reconstruction in $\mathcal{O}(\log n)$ rounds.

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RSBM is $(2n, d, b)$-clustered regular with $G\big|_{V_1}$, $G\big|_{V_2}$ expanders w.h.p. $\Longrightarrow$
Cor. Strong reconstruction $(a - b > 2\sqrt{d-1})$

Regular Clustered and Clustered Graphs

$(2n, d, b, \gamma)$**-clustered Graph.**

A graph $G = (V_1 \dot{\bigcup} V_2, E)$ s.t.

- $|V_1| = |V_2|$,
- every node has degree $d \pm \gamma d$
- each $v \in V_i$ has $b \pm \gamma d$ neighbors in V_{3-i}.

Thm. If $\min\{\lambda_2, \frac{a-b}{d}\} > \lambda$ and $\gamma = \mathcal{O}(\frac{a-b}{d} - \lambda_3)$
$\implies \mathcal{O}(\gamma^2/(\frac{a-b}{d} - \lambda_3)^2)$-weak reconstruction.

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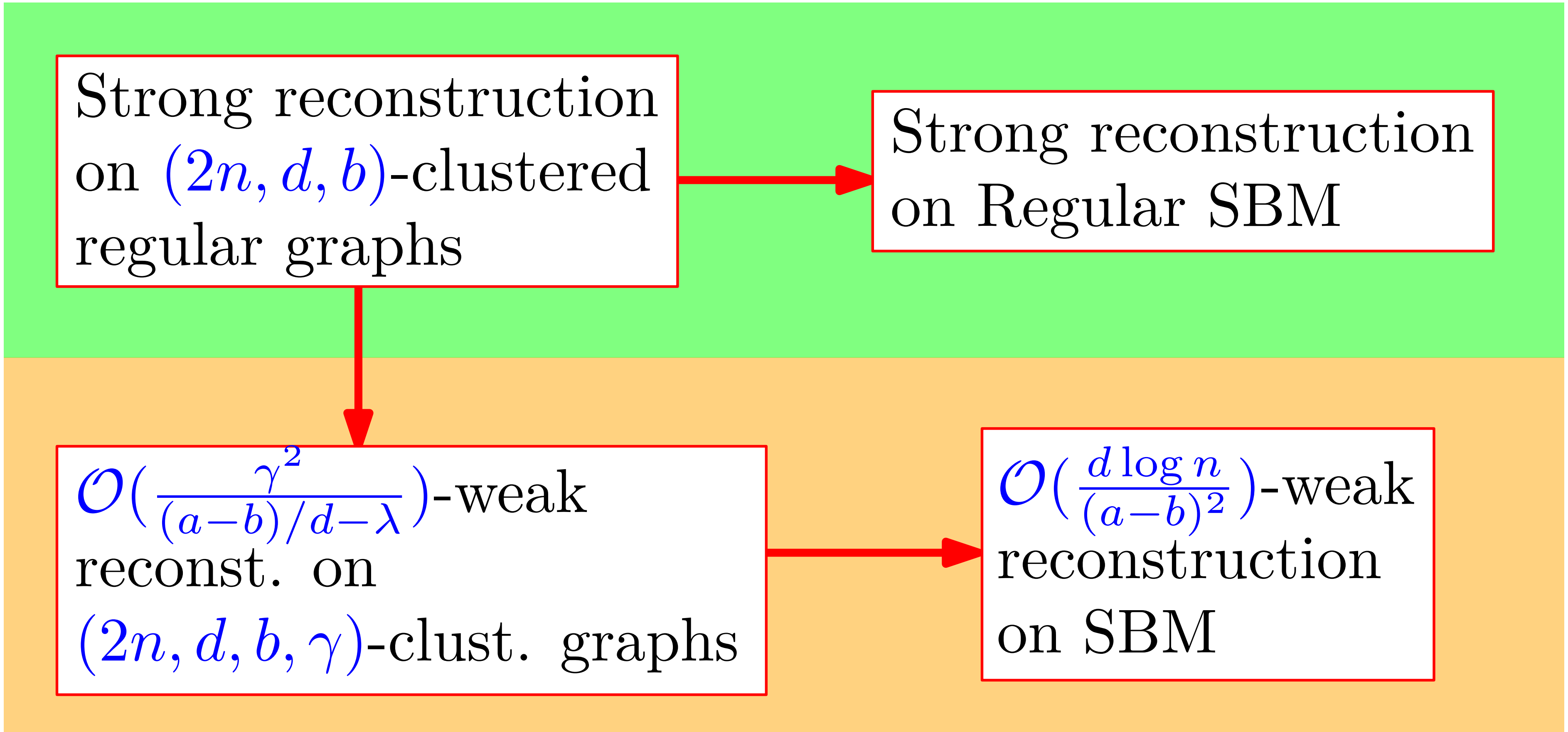
Cor. If $a - b > \sqrt{d \log n}$ and $b > \frac{\log n}{n^2}$, SBM is $(2n, d, b, 6\sqrt{\frac{\log n}{d}})$-clust. with $\min\{\lambda_2, 24\sqrt{\frac{\log n}{d}}\} > \lambda$ w.h.p. $\implies \mathcal{O}\left(\frac{d \log n}{(a-b)^2}\right)$-weak reconstruction.

Analysis: Roadmap



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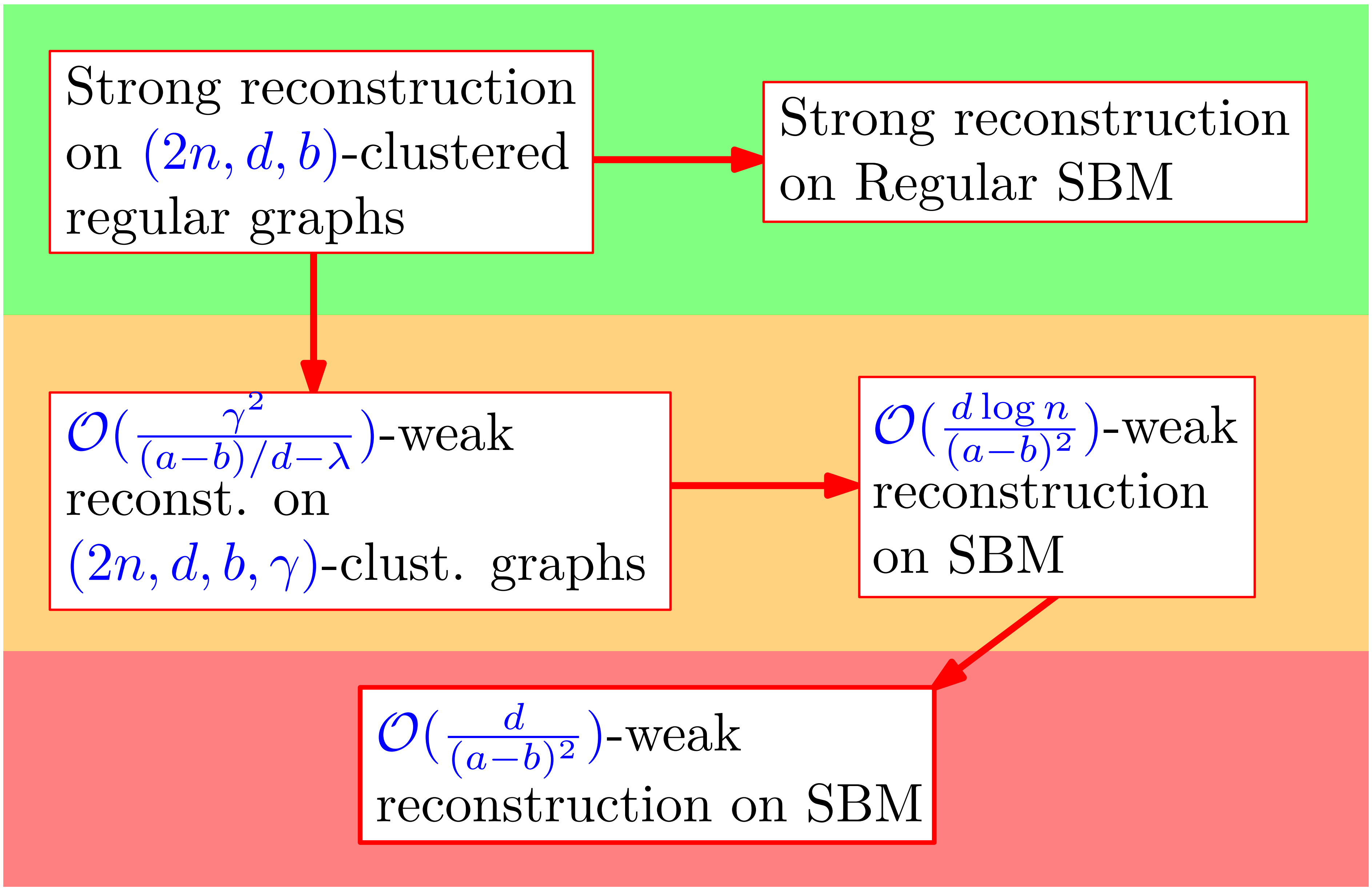

Analysis on Regular Graphs

$P = D^{-1}A = \frac{1}{d}A \longrightarrow$ symmetric $\implies$ orthonormal eigenvectors $\mathbf{v}_1, ..., \mathbf{v}_{2n}$ and real eigenvalues $\lambda_1, ..., \lambda_{2n}$.

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$$\mathbf{x}^{(t)} = P^t \cdot \mathbf{x}^{(0)} = \sum_i \lambda_i^t (\mathbf{v}_i^\intercal \mathbf{x}^{(0)}) \mathbf{v}_i \xrightarrow{t \to \infty} (\mathbf{v}_1^\intercal \mathbf{x}^{(0)}) \mathbf{v}_1$$

Perron-Frobenius Theorem:
$\lambda_1 = 1$, $|\lambda_{i \neq 1}| < 1$

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$$\mathbf{v}_1 = \frac{1}{\sqrt{2n}}\mathbf{1}$$

Regular clustered graphs $\implies$ $P\chi = \left(\frac{a-b}{d}\right) \cdot \chi$

Analysis on Regular Graphs

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$$\mathbf{x}^{(t)} = P^t \cdot \mathbf{x}^{(0)} = \textstyle\sum_i \lambda_i^t (\mathbf{v}_i^\intercal \mathbf{x}^{(0)}) \mathbf{v}_i$$

$$\mathbf{v}_1 = \frac{1}{\sqrt{2n}}\mathbf{1}$$

Regular clustered graphs $\Longrightarrow$ $P\chi = (\frac{a-b}{d}) \cdot \chi$

$$\frac{1}{d}\left(\begin{array}{c:c} \cdots\cdots\cdots & \cdots\cdots\cdots \\ \cdots a \text{ “1”s} \cdots & \cdots b \text{ “1”s} \cdots \\ \cdots\cdots\cdots & \cdots\cdots\cdots \\ \hdashline \cdots\cdots\cdots & \cdots\cdots\cdots \\ \cdots b \text{ “1”s} \cdots & \cdots a \text{ “1”s} \cdots \\ \cdots\cdots\cdots & \cdots\cdots\cdots \end{array}\right) \cdot \begin{pmatrix} 1 \\ \vdots \\ 1 \\ -1 \\ \vdots \\ -1 \end{pmatrix} = \frac{a-b}{d} \begin{pmatrix} 1 \\ \vdots \\ 1 \\ -1 \\ \vdots \\ -1 \end{pmatrix}$$

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Regular clustered graphs $\implies$ $P\chi = \left(\frac{a-b}{d}\right) \cdot \chi$

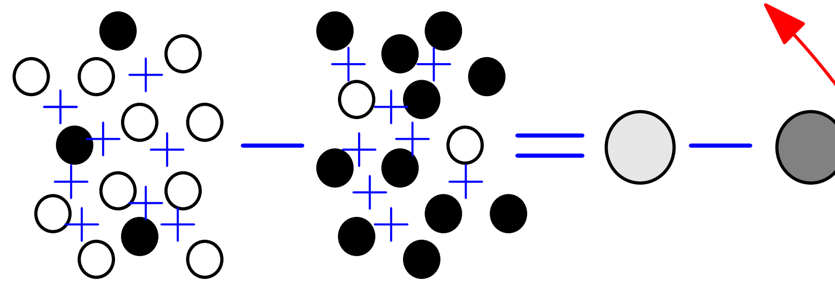
If $\lambda < \frac{a-b}{d} = \lambda_2$ then

$$\mathbf{x}^{(t+1)} = \frac{1}{2n}(\mathbf{1}^\intercal \mathbf{x}^{(0)})\mathbf{1} + \lambda_2^t \frac{1}{2n}(\chi^\intercal \mathbf{x}^{(0)})\chi + \mathbf{e}^{(t)}$$

with $\|\mathbf{e}^{(t)}\| = \left\| \sum_{i=3}^{2n} \lambda_i^t (\mathbf{v}_i^\intercal \mathbf{x}^{(0)}) \mathbf{v}_i \right\| \leq \lambda^t \|\mathbf{x}^{(0)}\| \leq \lambda^t \sqrt{2n}$

Analysis on Regular Graphs

$$\frac{1}{2}\left(\frac{1}{n}\sum_{u\in V_1}\mathbf{x}^{(0)}(u)-\frac{1}{n}\sum_{u\in V_2}\mathbf{x}^{(0)}(u)\right)$$



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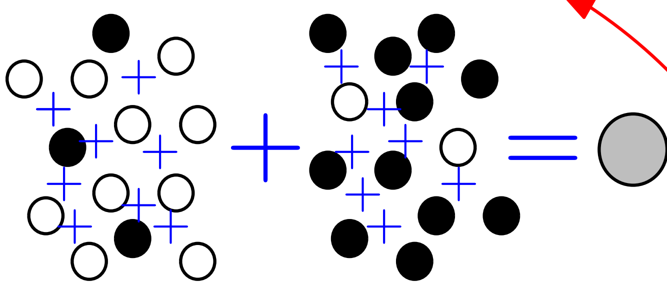

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Analysis on Regular Graphs

If $\boxed{\lambda(1+\delta) < \frac{a-b}{d} = \lambda_2}$ then

$$\mathbf{x}^{(t)} = \frac{1}{2n}(\mathbf{1}^\intercal \mathbf{x}^{(0)})\mathbf{1} + \lambda_2^t \frac{1}{2n}(\chi^\intercal \mathbf{x}^{(0)})\chi + \mathbf{e}^{(t)}$$

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$$\mathbf{x}^{(t)} - \mathbf{x}^{(t-1)} = (\chi^\intercal \mathbf{x}^{(0)})\lambda_2^{t-1}(\lambda_2 - 1)\chi + \underbrace{\mathbf{e}^{(t)} - \mathbf{e}^{(t-1)}}_{\ll \lambda_2^{t-1} \text{ if } t = \Omega(\log n)}$$

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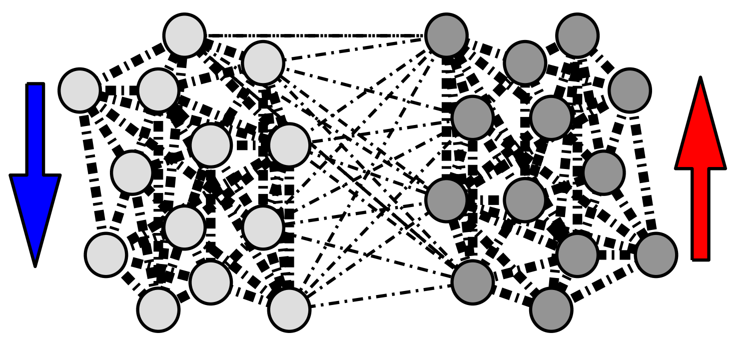
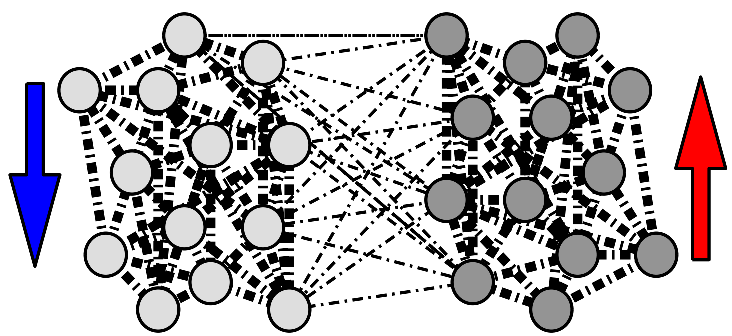


$$\boxed{\mathrm{sign}(\mathbf{x}^{(t)}(u) - \mathbf{x}^{(t-1)}(u)) \propto \mathrm{sign}(\chi(u))}$$

Analysis on Regular Graphs

Corollary.
RSBM is $(2n, d, b)$-clust. regular and
$\lambda = \mathcal{O}(\frac{1}{\sqrt{d}}) \ll \frac{a-b}{d}$ by *random degree* k *lifts*
[Friedman & Kohler]
$\implies$ Strong reconstruction in $\log n$ w.h.p.



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More Communities

(k, n, d, b)**-clustered Regular Graph.** A graph $G = (\dot{\bigcup}_{i=1}^{k} V_i, E)$ s.t.

- $|V_1| = \cdots = |V_k|$,
- every node has degree $d = a + (k-1)b$
- each $v \in V_i$ has b neighbors in V_j for $j \neq i$.

$\frac{a-b}{d}$ eigenval. with $\mathbf{v}_2, ..., \mathbf{v}_k$ eigenvec. s.t. constant on each V_i and $\mathbf{1}^\intercal \mathbf{v}_i = 0$.

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$(\frac{a-b}{d} = \lambda_2 = ... = \lambda_k)$

Thm. If $\frac{a-b}{d} > \lambda(1+\delta)$ with $\lambda = \max\{\lambda_{k+1}, |\lambda_{kn}|\}$, then $\Theta(\log n)$ parallel run of averaging gives strong reconstruction in $\mathcal{O}(\log n)$ rounds.

Future Work

Non-regular SBM.
How much “weak” with many communities?

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Sparisification.
At each round, pick an edge u.a.r. (*population protocols*):
those two nodes averages their values.
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Planted Clique.
$G_{n,p} \cup$ "clique of $\sqrt{n}(1+\delta)$ nodes":
Does averaging identify the clique?

Thank You!

Analysis of Clustered Graphs

$(2n, d, b, \gamma)$-clustered: degree $\pm\gamma d$

fundamental mapping

$\mathbf{v}$ eigenvector of $P = D^{-1}A \implies \mathbf{w} = D^{\frac{1}{2}}\mathbf{v}$
eigenvector of $N = D^{-\frac{1}{2}}AD^{-\frac{1}{2}} = D^{\frac{1}{2}}PD^{-\frac{1}{2}}$

spectral theorem: eigenvectors $\mathbf{w}_1, \ldots, \mathbf{w}_{2n}$.

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$$\mathbf{x}^{(t)} = P^t \cdot \mathbf{x}^{(0)} = D^{-\frac{1}{2}}N^t D^{\frac{1}{2}} \cdot \mathbf{x}^{(0)} =$$
$$D^{-\frac{1}{2}}N^t \cdot \left(D^{\frac{1}{2}}\mathbf{x}^{(0)}\right) = D^{-\frac{1}{2}}\sum_i \lambda_i^t(\mathbf{w}_i^\intercal D^{\frac{1}{2}}\mathbf{x}^{(0)})\mathbf{w}_i =$$
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$P\chi \nparallel \chi$ but...

Analysis of Clustered Graphs

$$\mathbf{x}^{(t)} = (\mathbf{w}_1^\intercal D^{\frac{1}{2}} \mathbf{x}^{(0)})\mathbf{v}_1 + \lambda_2^t \underbrace{\left(\frac{\mathbf{w}_2^\intercal D^{\frac{1}{2}} \mathbf{x}^{(0)}}{\mathbf{w}_2^\intercal D^{\frac{1}{2}} \chi} \right)}_{=\beta} (\chi + \mathbf{z}) + \mathbf{e}^{(t)}$$

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$$= \frac{1}{\sum_u d_u} \left(\sum_{u \in V} d_u \mathbf{x}^{(0)}(u)\right)$$

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$$\begin{aligned}&\|\mathbf{z}\| \\ &= \|(\mathbf{w}_2^\intercal D^{\frac{1}{2}}\chi)\mathbf{v}_2 - \chi\| \\ &= \|D^{-\frac{1}{2}}((\mathbf{w}_2^\intercal D^{\frac{1}{2}}\chi)\mathbf{w}_2 - D^{\frac{1}{2}}\chi)\| \\ &\leq \frac{88\gamma}{\frac{a-b}{d} - \lambda_3}\sqrt{2n}\end{aligned}$$

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$$\sum_{i=3}^{2n} \lambda_i^t(\mathbf{w}_i^\intercal D^{\frac{1}{2}}\mathbf{x}^{(0)})\mathbf{v}_i \leq 4\lambda^t\sqrt{2n}$$

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$$\mathbf{x}^{(t)} = (\mathbf{w}_1^\intercal D^{\frac{1}{2}} \mathbf{x}^{(0)})\mathbf{v}_1 + \lambda_2^t \underbrace{\left(\frac{\mathbf{w}_2^\intercal D^{\frac{1}{2}} \mathbf{x}^{(0)}}{\mathbf{w}_2^\intercal D^{\frac{1}{2}} \chi} \right)}_{=\beta} (\chi + \mathbf{z}) + \mathbf{e}^{(t)}$$

$$\|\mathbf{z}\| \leq \frac{\gamma}{\frac{a-b}{d} - \lambda_3} \cdot 88\sqrt{2n} \implies$$

$$\left|\left\{u \in V \mid \mathbf{z}(u) \geq \tfrac{1}{2}\right\}\right| = \mathcal{O}\left(\frac{\gamma^2}{(\frac{a-b}{d} - \lambda_3)^2} \cdot n \right)$$

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$$\mathbf{x}^{(t)} - \mathbf{x}^{(t-1)} = \beta\lambda_2^{t-1}(\lambda_2 - 1)(\chi + \mathbf{z}) + \underbrace{\mathbf{e}^{(t)} - \mathbf{e}^{(t-1)}}_{\ll \lambda_2^{t-1} \text{ if } t = \Omega(\log n) \text{ and } \lambda < \lambda_2(1+\delta)}$$

$\gg n^{-const}$

$$\mathbf{z}(u) \leq \frac{1}{2} \implies$$

$$\mathrm{sign}(\mathbf{x}^{(t)}(u) - \mathbf{x}^{(t-1)}(u)) = \mathrm{sign}(\chi(u)) \text{ or } -\mathrm{sign}(\chi(u))$$

Analysis of Clustered Graphs

Thm. G $(2n, d, b, \gamma)$-clustered. If $\lambda_2 > \lambda(1+\delta)$, $\frac{a-b}{d} > \lambda$ and $\gamma = \mathcal{O}(\frac{a-b}{d} - \lambda_3)$ then in $\mathcal{O}(\log n)$ rounds averaging gives $\mathcal{O}\left(\frac{\gamma^2}{(\frac{a-b}{d}-\lambda_3)^2}\right)$-weak reconstrunction.

Cor. G $(2n, d, b, \gamma)$-clustered.
If $\frac{a-b}{d} - 10\gamma > \lambda(1+\delta)$ then in $\mathcal{O}(\log n)$ rounds averaging gives $\mathcal{O}\left(\frac{\gamma^2}{(\frac{a-b}{d}-\lambda_3)^2}\right)$-weak reconstrunction.

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Lem. $G \sim \mathcal{G}_{2n,\frac{a}{n},\frac{b}{n}}$. If $a - b > \sqrt{d \log n}$ then w.h.p.

i) G is $(2n, d, b, 6\sqrt{\frac{\log n}{d}})$-clustered and

ii) $\lambda \leq \min\left\{\frac{\lambda_2}{1+\delta}, 24\sqrt{\frac{\log n}{d}}\right\}$.

Cor. $G \sim \mathcal{G}_{2n,\frac{a}{n},\frac{b}{n}}$. If $a - b > 25\sqrt{d \log n}$ and $b = \Omega(\log n/n^2)$ then averaging gives $\mathcal{O}\left(\frac{d \log n}{(a-b)^2}\right)$-weak reconstrunction in $\mathcal{O}(\log n)$ rounds w.h.p.

Technical Lemmas

Lem. $G \sim \mathcal{G}_{2n,\frac{a}{n},\frac{b}{n}}$. If $a - b > \sqrt{d}\log n$ then w.h.p.

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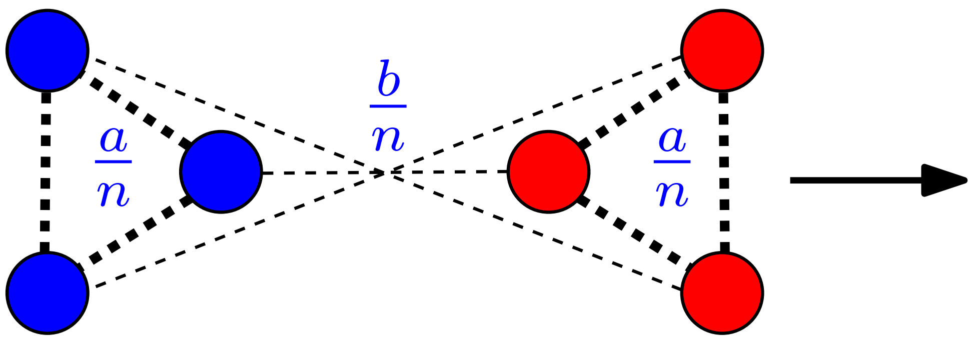
Lem. $G \sim \mathcal{G}_{2n,\frac{a}{n},\frac{b}{n}}$. If $d > \log n$ then w.h.p.

$$\|\tfrac{1}{d}A - \tfrac{1}{d}\mathbb{E}[A]\| \leq \mathcal{O}\left(\sqrt{\frac{\log n}{d}}\right).$$

Lem. G $(2n, d, b, \gamma)$-clustered. If $\frac{a-b}{d} > 12\gamma$ and $\|\frac{1}{d}A - \frac{1}{d}\text{“}\mathbb{E}[A]\text{”}\| \leq \gamma$ then for $3 \leq i \leq 2n$ we have $|\lambda_i| \leq 4\gamma$ and $\lambda_2 \geq \lambda_3(1+\delta)$.

Spectral Technique for SBM



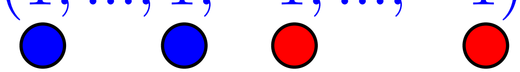
$$A = (\mathbb{1}_{((u,v)\in E)})_{u,v\in V}$$

$$\mathbb{E}[A] = \left(\begin{array}{ccc:ccc}
\frac{a}{n} & \cdots & \frac{a}{n} & \frac{b}{n} & \cdots & \frac{b}{n} \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
\frac{a}{n} & \cdots & \frac{a}{n} & \frac{b}{n} & \cdots & \frac{b}{n} \\
\hdashline
\frac{b}{n} & \cdots & \frac{b}{n} & \frac{a}{n} & \cdots & \frac{a}{n} \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
\frac{b}{n} & \cdots & \frac{b}{n} & \frac{a}{n} & \cdots & \frac{a}{n}
\end{array}\right)$$

$v_1, \ldots, v_n$ eigenvectors of $\mathbb{E}[A]$:

$$v_1 = \mathbb{1} = (1, \ldots, 1)$$

$$v_2 = \chi = (1, \ldots, 1, -1, \ldots, -1)$$



Concentration on Expectation

Theorem. $X = \sum X_i$ independent binary r.v.s with $|X_i - \mathbb{E}[X_i]| \leq M$, then

$$\Pr(\|X - \mathbb{E}[X]\| \geq t) \leq 2e^{\frac{-t^2}{2\left(\sum \mathrm{Var}[X_i] + \frac{M}{3}\lambda\right)}}.$$

Theorem (Matrix Bernstein Inequality). $X_1, \ldots, X_N$ independent $n \times n$ symmetric random matrices with $\|X_i - \mathbb{E}[X_i]\| \leq L$ for some L. Let $\sigma^2 := \|\sum_i (\mathbb{E}[X_i^2] - \mathbb{E}[X_i]^2)\|$. For every t, we have

$$\Pr\left(\|\sum_i (X_i - \mathbb{E}[X_i])\| \geq t\right) \leq 2ne^{\frac{-t^2}{2\left(\sigma^2 + \frac{L}{3}t\right)}}.$$

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$$X_{i,j} = (A - \mathbb{E}[A])_{i,j}$$

$$t = \sqrt{d \log n}$$

Lem. $G \sim \mathcal{G}_{2n, \frac{a}{n}, \frac{b}{n}}$. If $d > \log n$ then w.h.p.

$$\|A - \mathbb{E}[A]\| \le \mathcal{O}(\sqrt{d \log n}).$$

Eigenvalue Perturbation

Thm (Weyl). M_1 and M_2 two Hermitian matrices, let $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ and $\lambda'_1 \geq \lambda'_2 \geq \cdots \geq \lambda'_n$ be the eigenvalues of M_1 and M_2 with multiplicities. For every i

$$|\lambda_i - \lambda'_i| \leq \|M_1 - M_2\|.$$

Lem. G $(2n, d, b, \gamma)$-clustered. If $\frac{a-b}{d} > 12\gamma$ and $\|\frac{1}{d}A - \frac{1}{d}\mathbb{E}[A]\| \leq \gamma$ then for $3 \leq i \leq 2n$ we have $|\lambda_i| \leq 4\gamma$ and $\lambda_2 \geq \lambda_3(1+\delta)$.

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$$|\lambda_i - \lambda_i'| \leq \|M_1 - M_2\|.$$

$$\frac{1}{d}\mathbb{E}[A] \cdot \mathbf{1} = \mathbf{1}$$
$$\frac{1}{d}\mathbb{E}[A] \cdot \chi = \frac{a-b}{d}\chi$$
$$\frac{1}{d}\mathbb{E}[A] \cdot \mathbf{v} = 0 \text{ if } v \perp \mathbf{1}, \chi$$

$$N = D^{-\frac{1}{2}} A D^{-\frac{1}{2}} \qquad \frac{1}{d}\mathbb{E}[A]$$

$$\|N - \tfrac{1}{d}\mathbb{E}[A]\| \leq 4\gamma$$

Lem. G $(2n, d, b, \gamma)$-clustered. If $\frac{a-b}{d} > 12\gamma$ and $\|\frac{1}{d}A - \frac{1}{d}\mathbb{E}[A]\| \leq \gamma$ then for $3 \leq i \leq 2n$ we have $|\lambda_i| \leq 4\gamma$ and $\lambda_2 \geq \lambda_3(1+\delta)$.

Better Bounds for the SBM

Cor. $G \sim \mathcal{G}_{2n,\frac{a}{n},\frac{b}{n}}$. If $a - b > 25\sqrt{d \log n}$ and $b = \Omega(\log n / n^2)$ then averaging gives $\mathcal{O}\left(\frac{d \log n}{(a-b)^2}\right)$-weak reconst. in $\mathcal{O}(\log n)$ rounds w.h.p.

Lem. $G \sim \mathcal{G}_{2n,\frac{a}{n},\frac{b}{n}}$. If $a - b > const \cdot \sqrt{d} > \sqrt{\log n}$ and $d < n^{\frac{1}{3}-\epsilon}$ then

- $\lambda_2 \geq \frac{a-b}{d} - \mathcal{O}(\frac{1}{\sqrt{d}})$,
- $\lambda_2 \geq \lambda(1+\delta)$,
- $|\sqrt{2nd} D^{-\frac{1}{2}} \mathbf{w}_2(i) - \chi(i)| \leq \frac{1}{100}$ for $i \in V/S$ for some S s.t. $|S| = \mathcal{O}(\frac{nd}{(a-b)^2})$.

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3. $|\sqrt{2nd}D^{-\frac{1}{2}}\mathbf{w}_2(i) - \chi(i)| \leq \frac{1}{100}$ for $i \in V/S$ for some S s.t. $|S| = \mathcal{O}(\frac{nd}{(a-b)^2})$.

Better Bounds for the SBM

Lem. $G \sim \mathcal{G}_{2n,\frac{a}{n},\frac{b}{n}}$. If $a-b > const \cdot \sqrt{d} > \sqrt{\log n}$ and $d < n^{\frac{1}{3}-\epsilon}$ then

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1 and 2 by Weyl's Thm. and $\|N - \frac{1}{d}\mathbb{E}[A]\| = \mathcal{O}(\frac{1}{\sqrt{d}})$.

$$\lesssim \underbrace{\|\frac{1}{d}A - \frac{1}{d}\mathbb{E}[A]\|}_{\leq \frac{1}{\sqrt{d}} \text{ by [Le \& Vershynin]}} + \|\frac{1}{d}\mathbb{E}[A](I - \frac{1}{\sqrt{d}}D^{\frac{1}{2}})\|$$

$$+ \|(I - \frac{1}{\sqrt{d}}D^{\frac{1}{2}})\frac{1}{d^{\frac{3}{2}}}\mathbb{E}[A]D^{\frac{1}{2}}\| = \mathcal{O}(\frac{1}{\sqrt{d}})$$

$\sum_{i \in V} |\sqrt{d} - \sqrt{d_i}|^2 \leq 2n$

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Theorem (Davis & Kahan, 1970). M_1 and M_2 symm. real matrices, $\mathbf{x}$ unit eigenvec. of M_1 of eigenval. t, $\mathbf{x}_p$ projection of $\mathbf{x}$ on eigenspaces of M_2 with eigenvalues $\leq t - \delta$. Then

$$\|\mathbf{x}_p\| \leq \frac{2}{\delta\pi}\|M_1 - M_2\|.$$

$\mathbf{w}_2 = \mathbf{w}^{(\mathbf{1})} + \mathbf{w}^{(\chi)} + \mathbf{w}^{(\perp)}$

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$t = \lambda_2$, $\mathbf{x} = \mathbf{w}_2$, $\delta = \frac{\lambda_2}{2}$

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$\|\mathbf{w}_2 - \frac{1}{\sqrt{2n}}\chi\|^2 = 2 - 2\|\mathbf{w}^{(\chi)}\|$
$\leq 2(\|\mathbf{w}^{(1)}\|^2 + \|\mathbf{w}^{(\perp)}\|^2)$
$\leq \mathcal{O}\left(\frac{d}{(a-b)^2}\right)$

$\|N - \frac{1}{d}\mathbb{E}[A]\| \leq \mathcal{O}(\frac{1}{\sqrt{d}})$

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