On the Necessary Memory to Compute the Plurality in Multi-Agent Systems

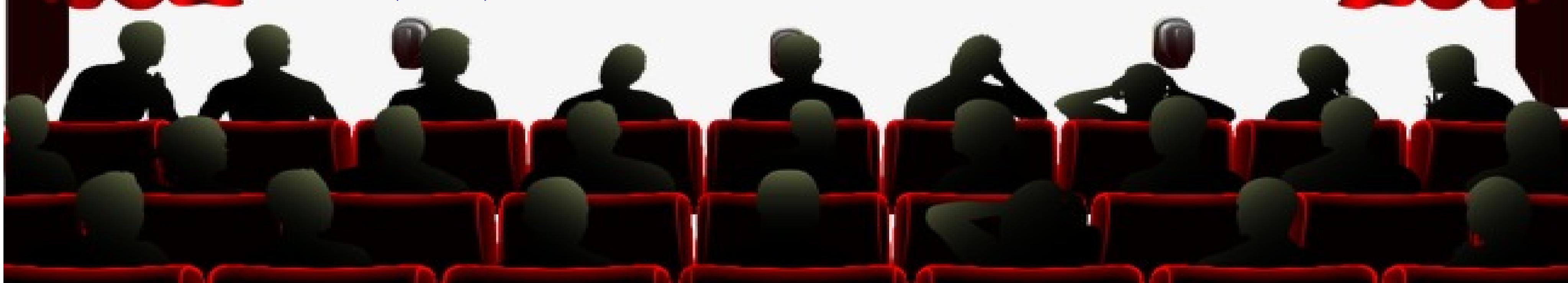
Emanuele Natale

joint work with
Iliad Ramezani (SUT, Iran)

Rome, 29 May 2019

Outline

- Problem: k-Plurality Consensus
- Model: Population Protocols
- Simple case: Majority Consensus
- Previous Work: $\Omega(2^k)$ Conjecture
- $\Omega(k^2)$ Lower Bound
- Previous Work: $O(k^6)$ *Almost* Refutation
- $O(k^{11})$ Upper Bound



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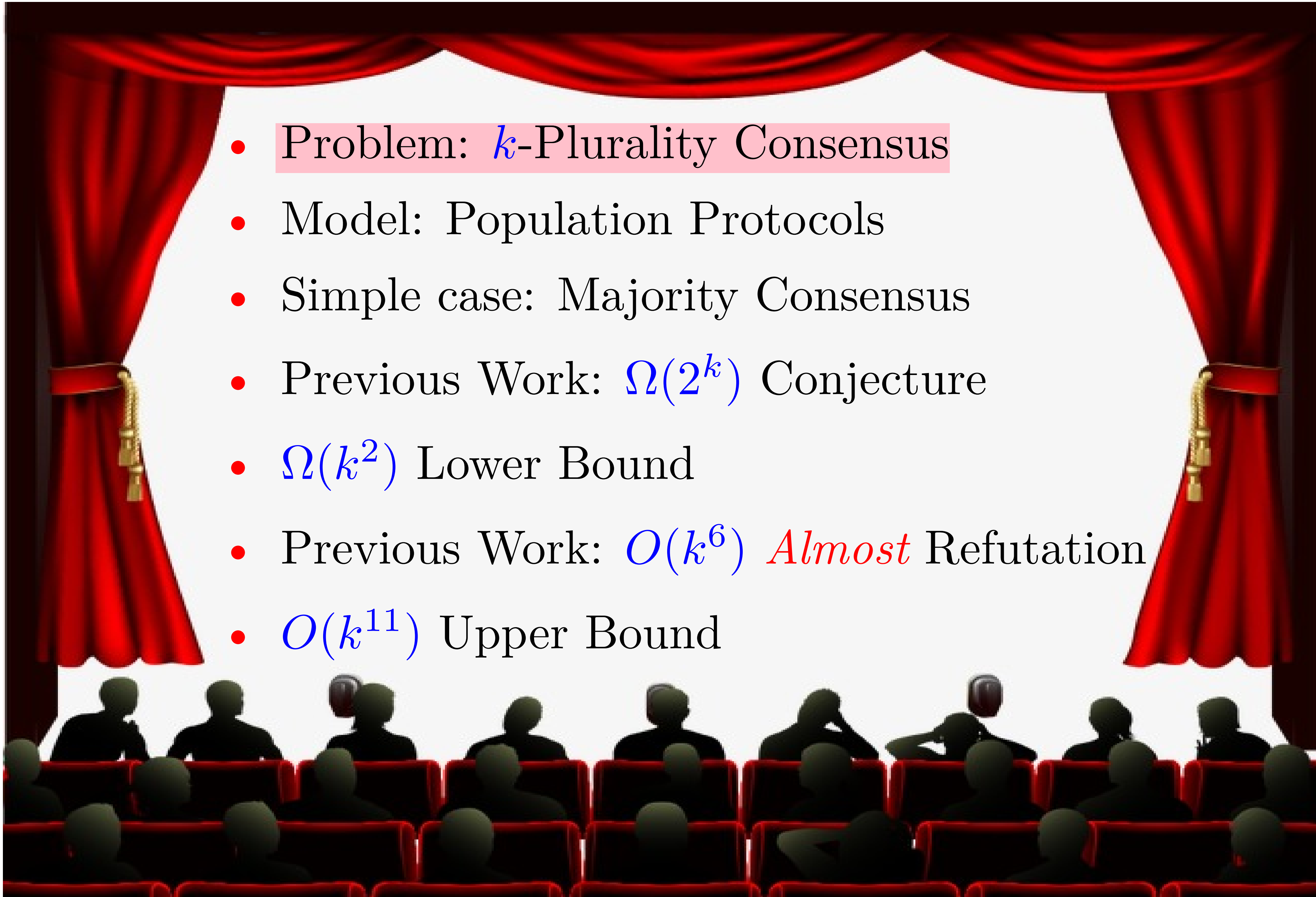

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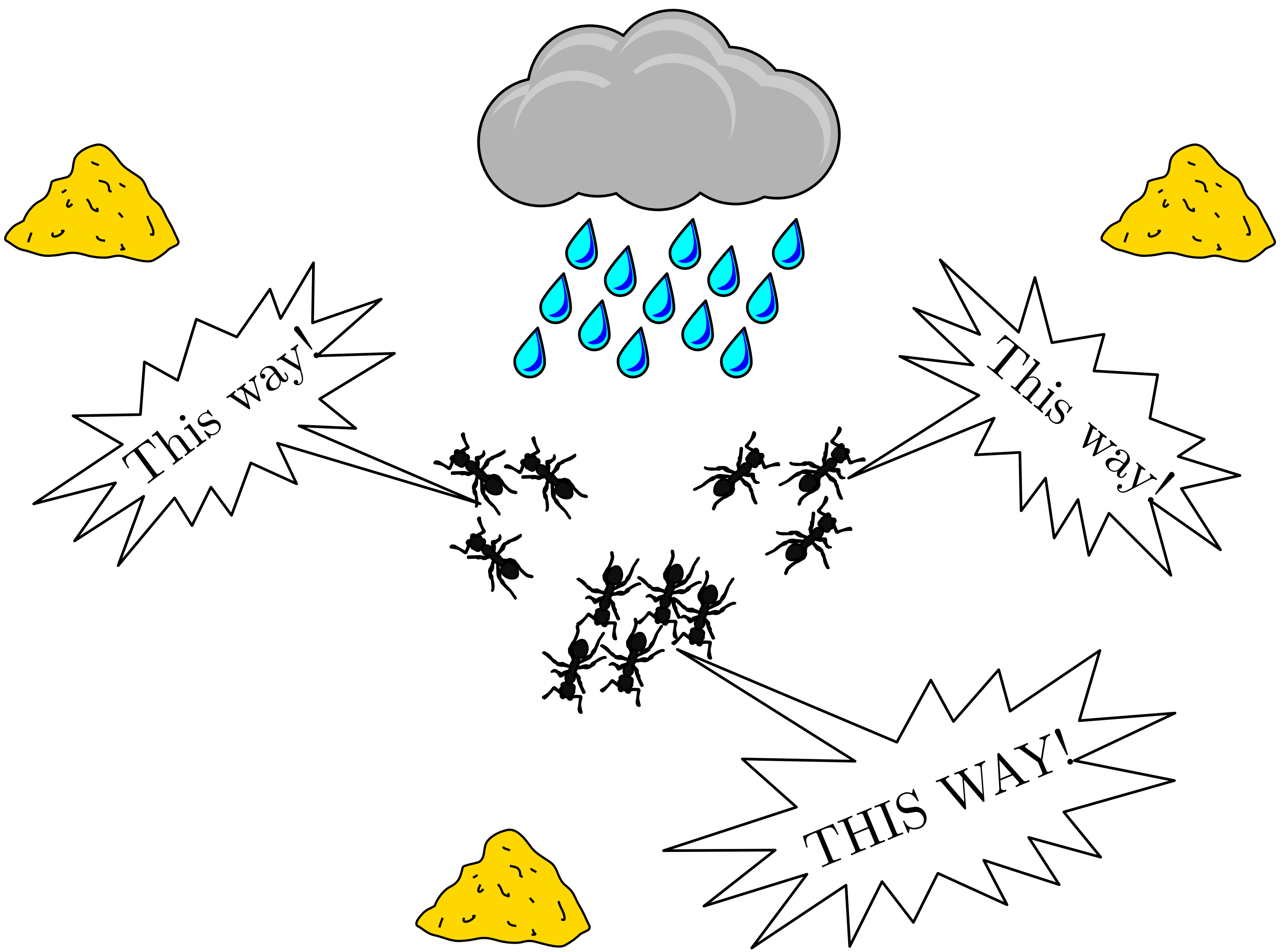
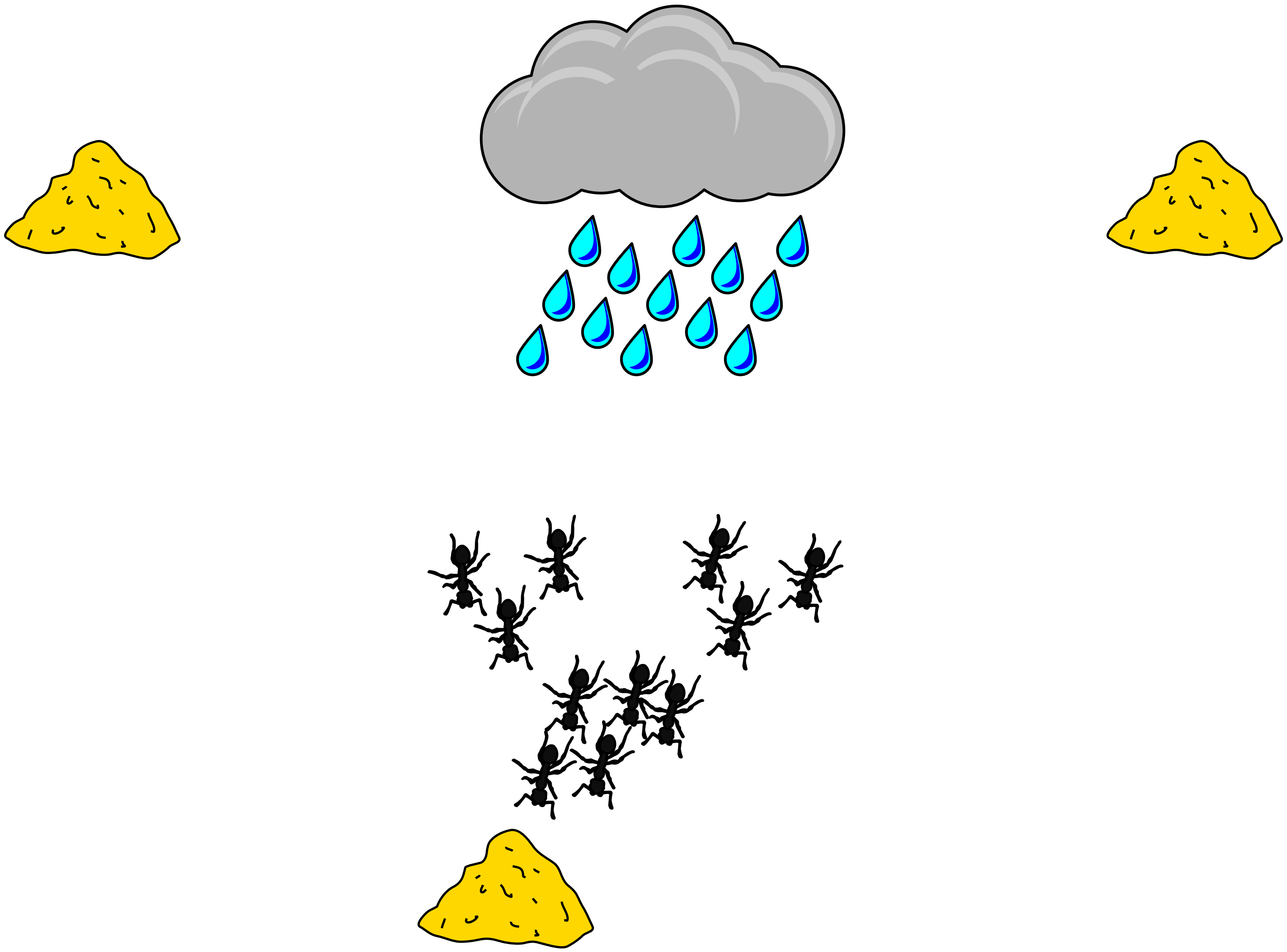
k-Plurality Consensus

Each agent supports one out of k opinions



k-Plurality Consensus

All agents eventually support the same opinion



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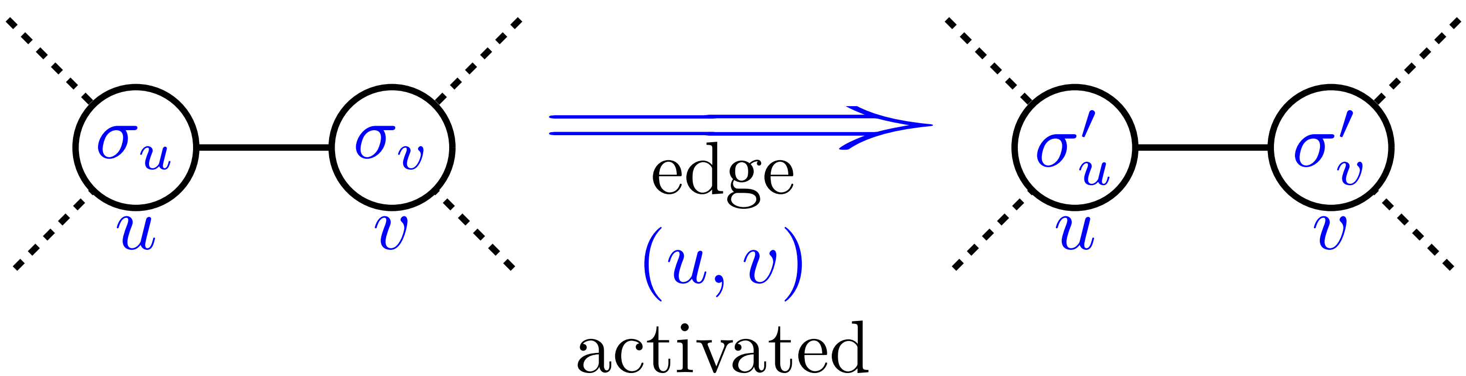
- edges activated by a *scheduler*,
- function $\gamma : \Sigma \times \Sigma \to \Sigma \times \Sigma$ s.t. if edge (u, v) with states (σ_u, σ_v) activated, new states are

$$\gamma(\sigma_u, \sigma_v) = (\sigma'_u, \sigma'_v)$$



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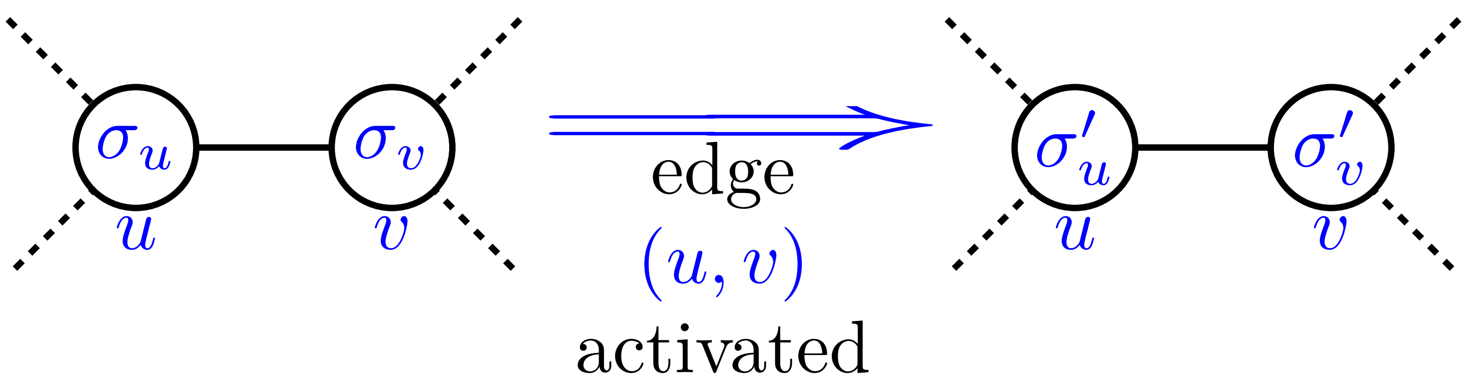
protocol's memory



Population Protocols: Schedulers

Probabilistic scheduler:
activate an edge chosen
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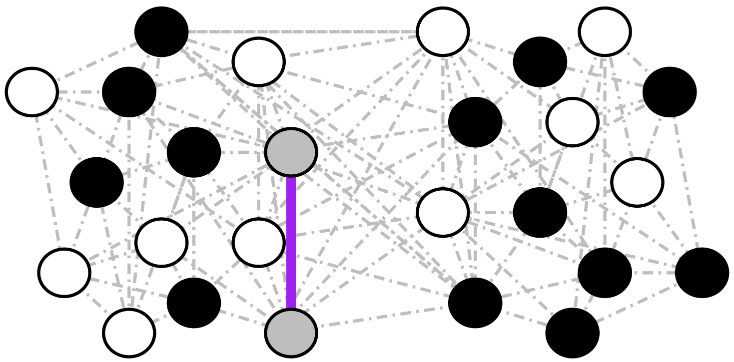

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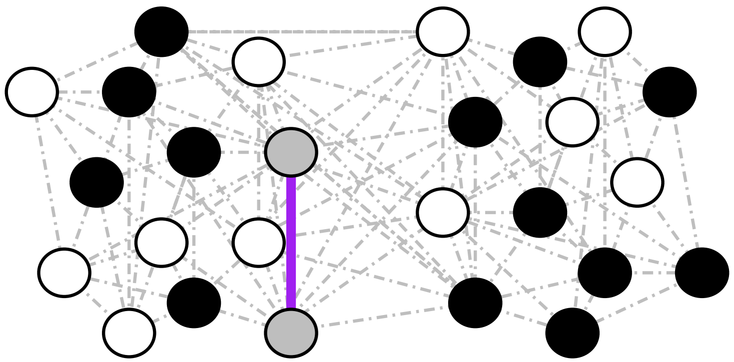

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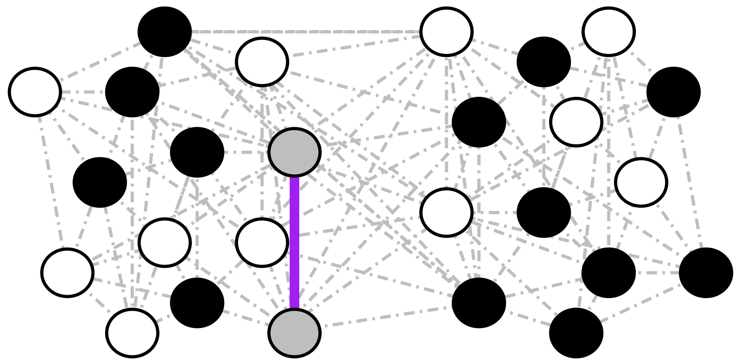

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A *configuration* is the state of all nodes $S = (\sigma_1, ..., \sigma_n)$.

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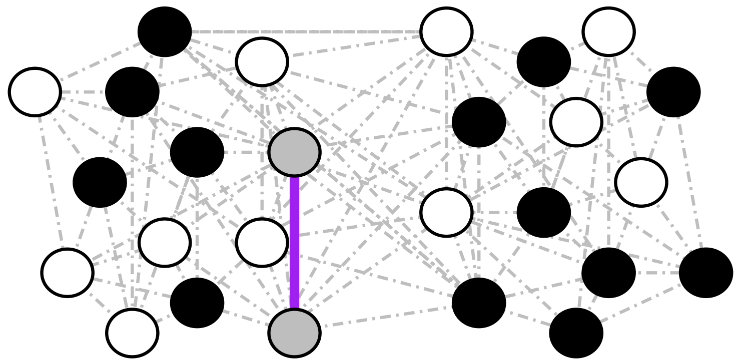

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Fair scheduler: if S appears infinitely often, also any conf. reachable from S appears infinitely often:

S' reachable from S and $S_1, S_2, ..., S, ..., S, ..., S, ...$

$\implies S_1, S_2, ..., S', ..., S', ..., S', ...$

Self-Stabilization

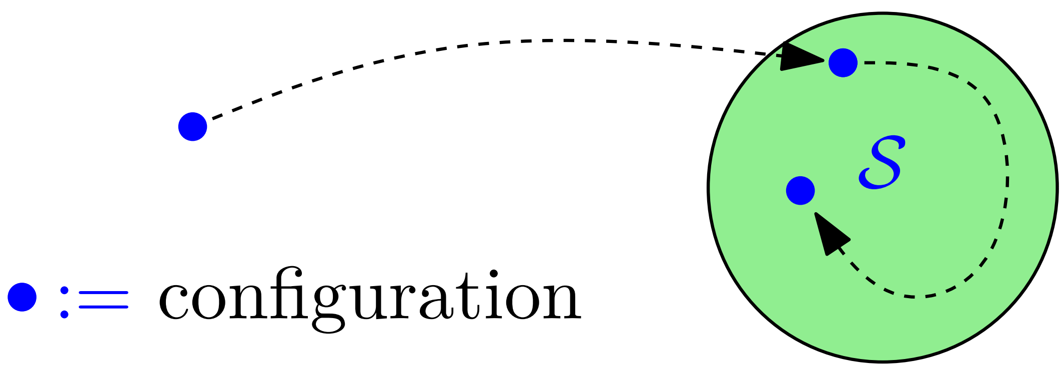
n agents with states in Σ. Σ^n possible configurations.

$\mathcal{S}$:= {"*correct* states of the system" }.

Convergence. Starting from any possible configuration, the system eventually reaches a configuration in $\mathcal{S}$.

Closure. If configuration in $\mathcal{S}$, it remains in $\mathcal{S}$.

A protocol is self-stabilizing iff guarantees convergence and closure w.r.t. $\mathcal{S}$.



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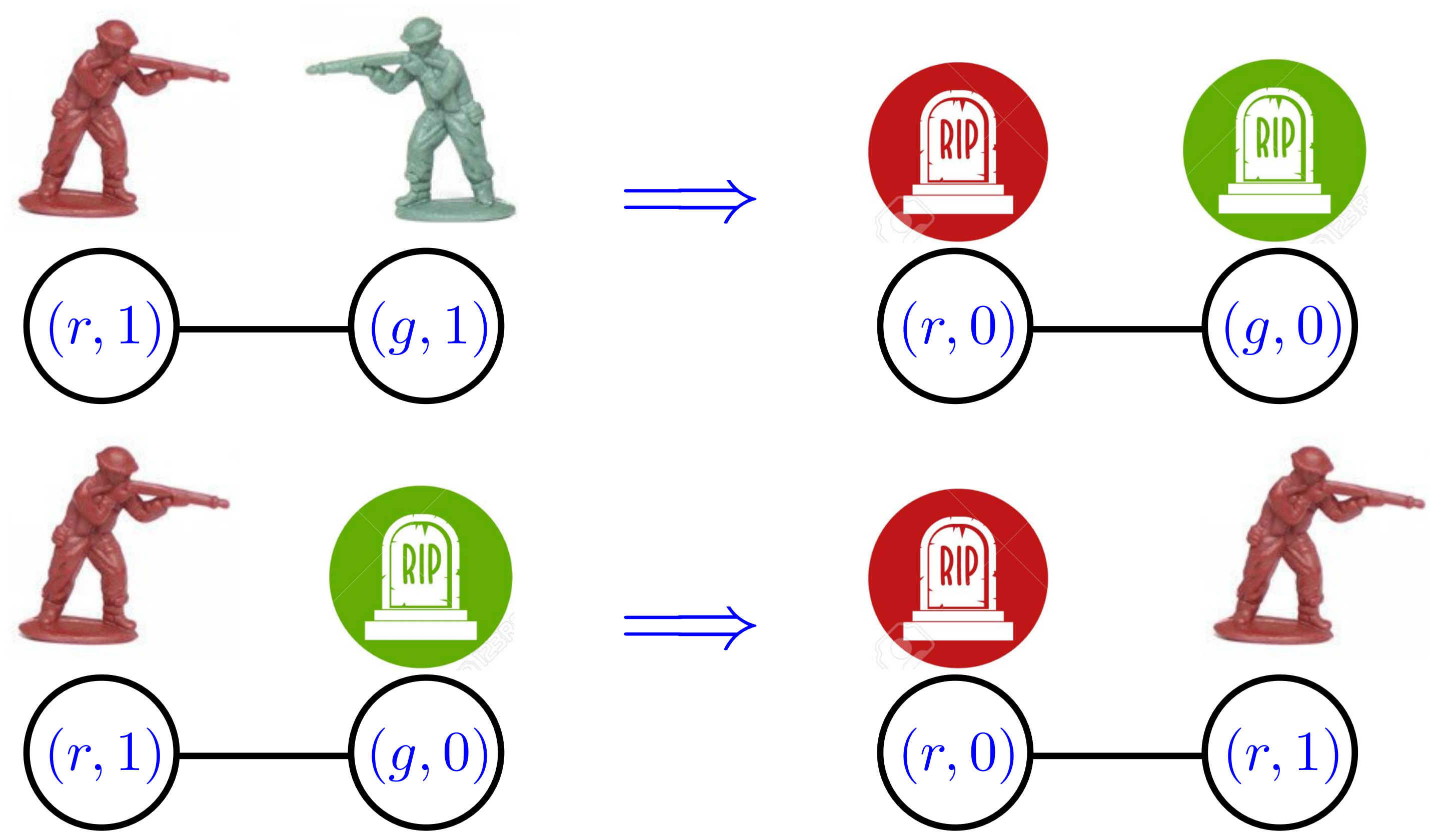
Majority (2-Plurality) Consensus: 2-bit Protocol

[Mertzios et al. ICALP'16, Benezit et al. ICASSP'09]

State: (green/red, defended or not)

$u \setminus v$	$(g,0)$	$(g,1)$	$(r,0)$	$(r,1)$
$(g,0)$	–	$((g,1),(g,0))$	–	$((r,1),(r,0))$
$(g,1)$	$((g,0),(g,1))$	–	$((g,0),(g,1))$	$((g,0),(r,0))$
$(r,0)$	–	$((g,1),(g,0))$	–	$((r,1),(r,0))$
$(r,1)$	$((r,0),(r,1))$	$((r,0),(g,0))$	$((r,0),(r,1))$	–



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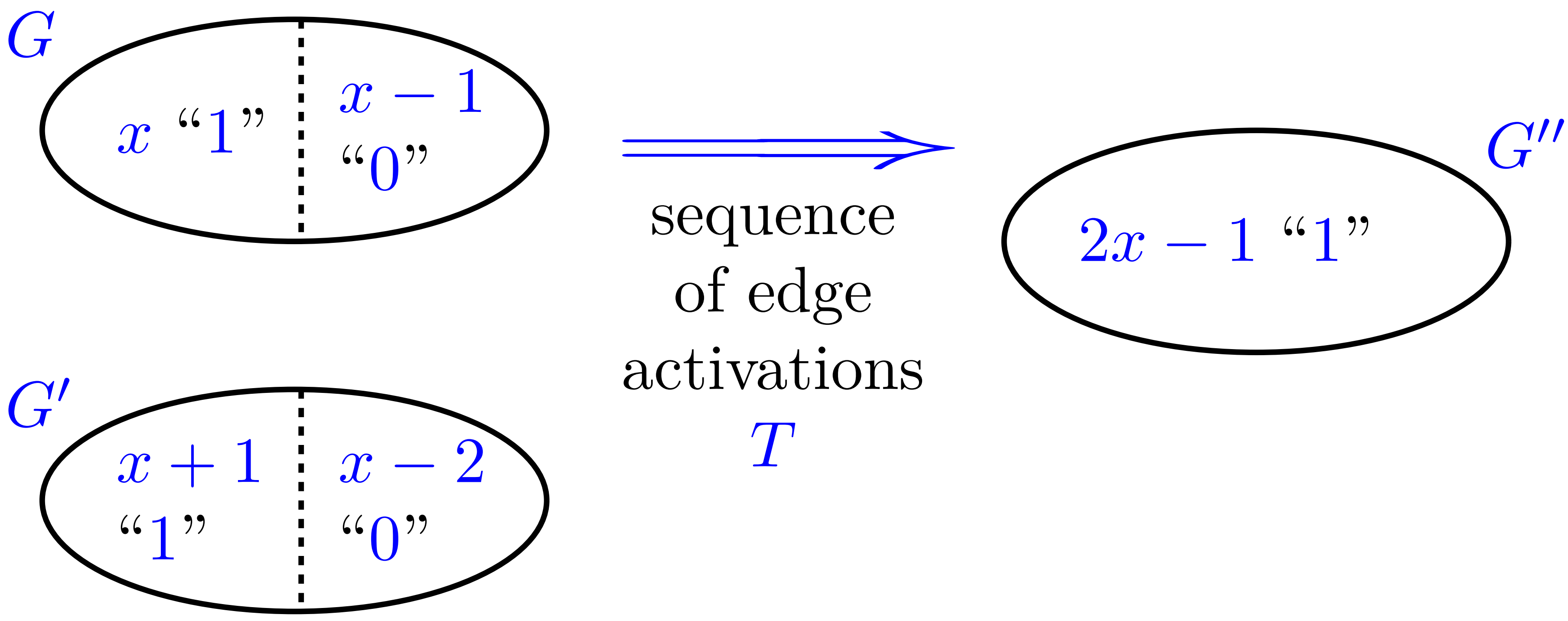

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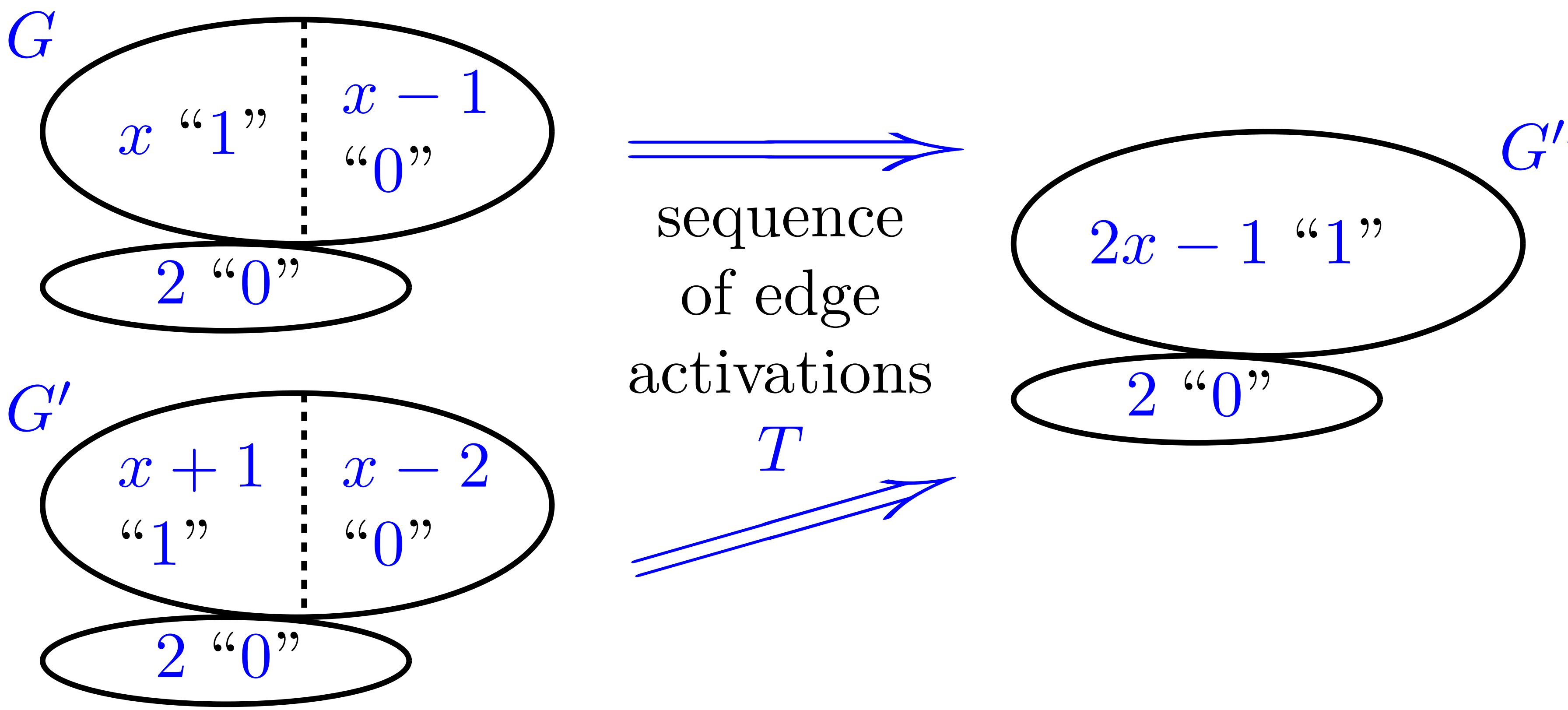

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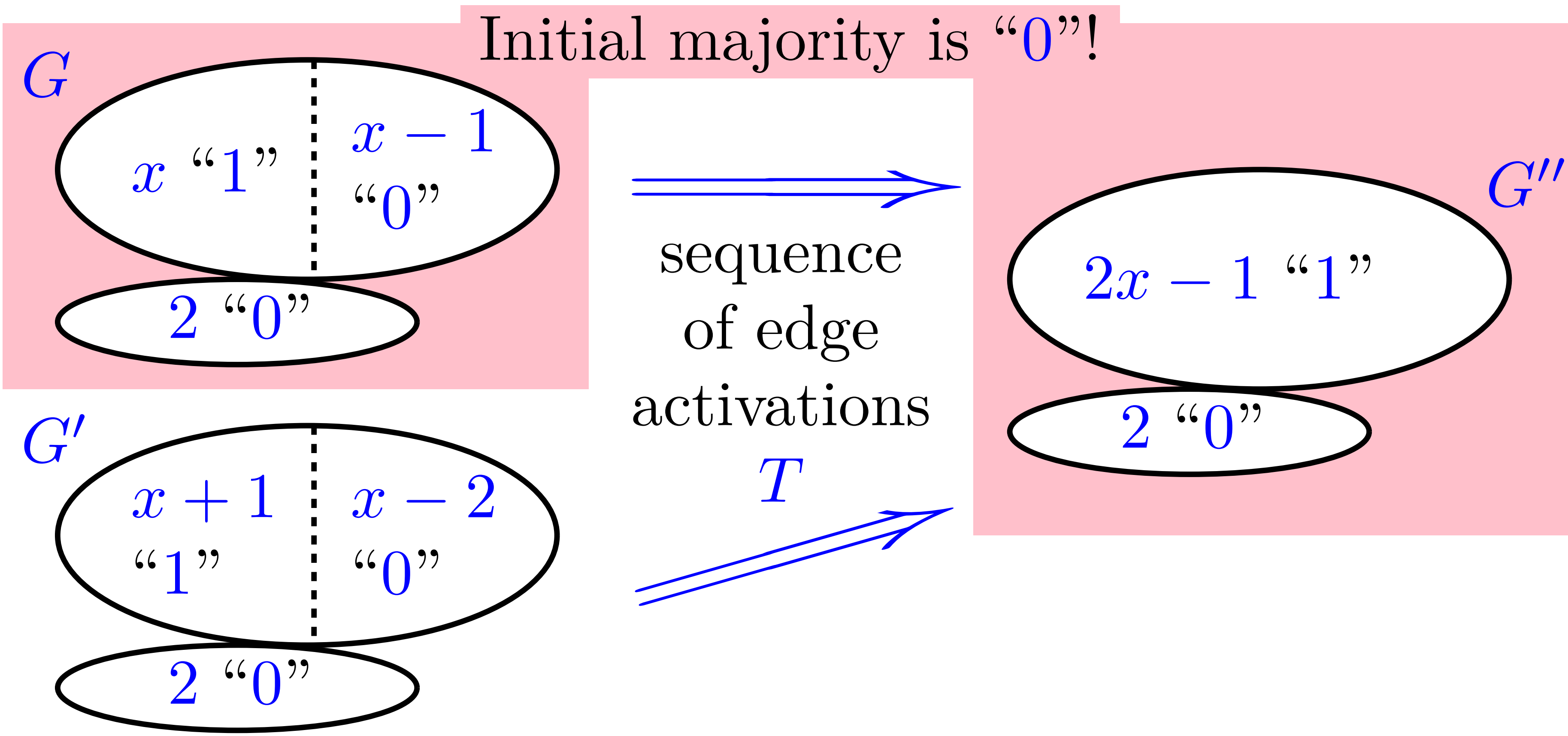

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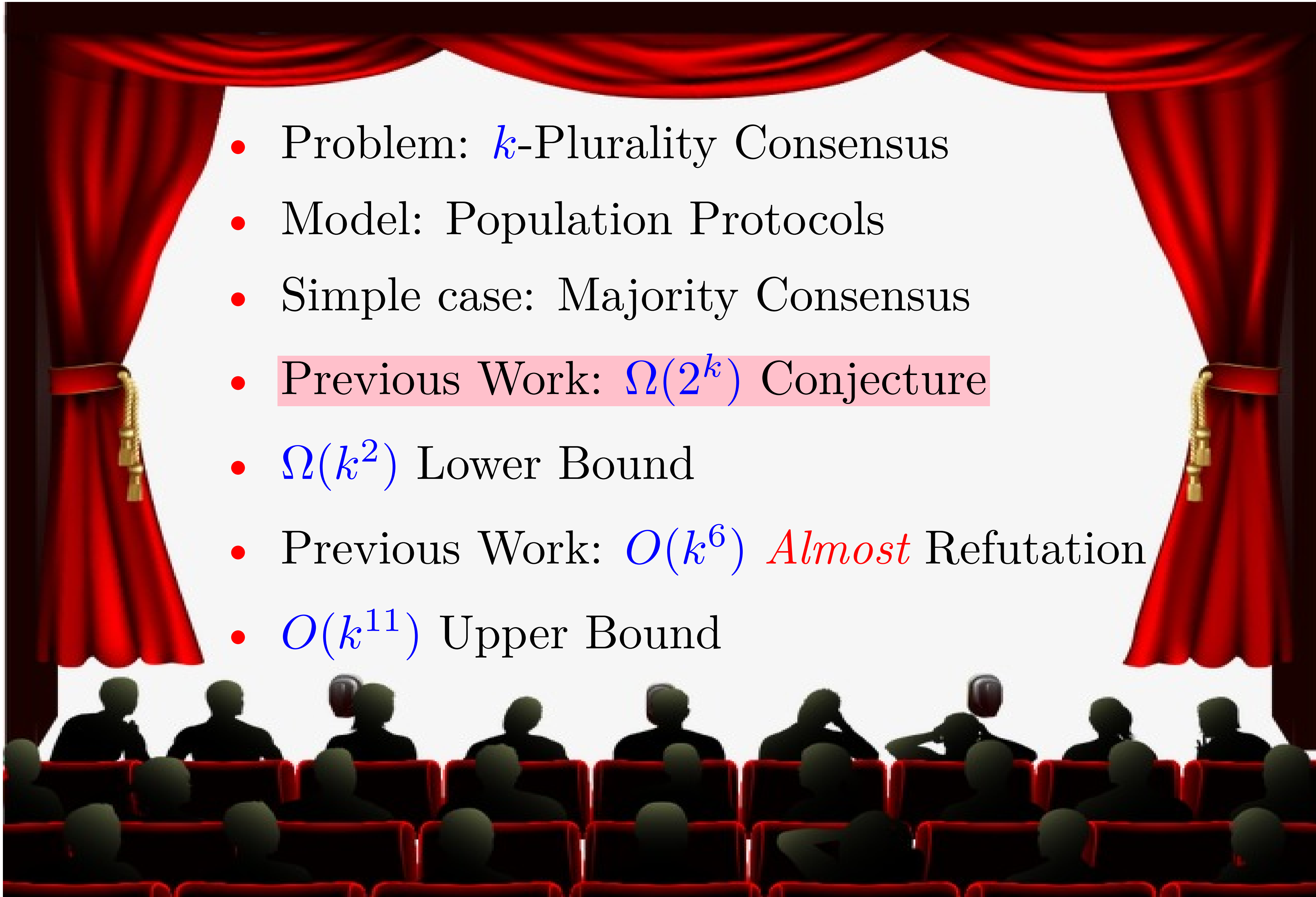

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Opinions can *only be tested for equality*.

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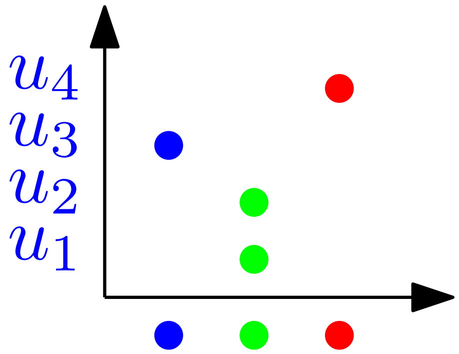

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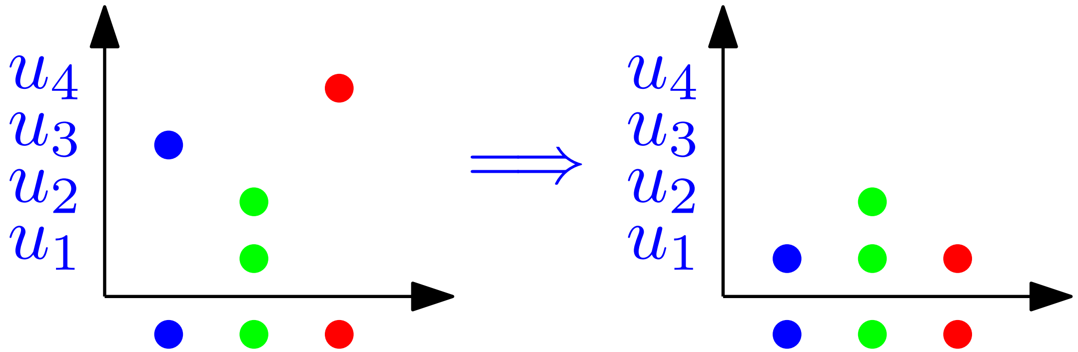

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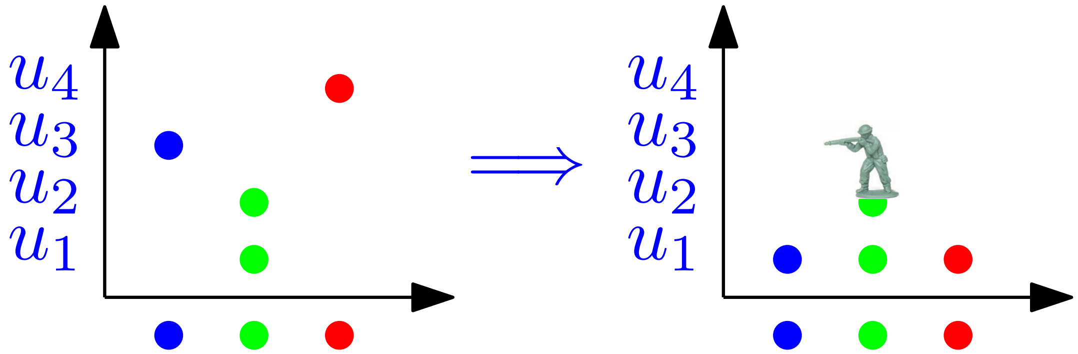

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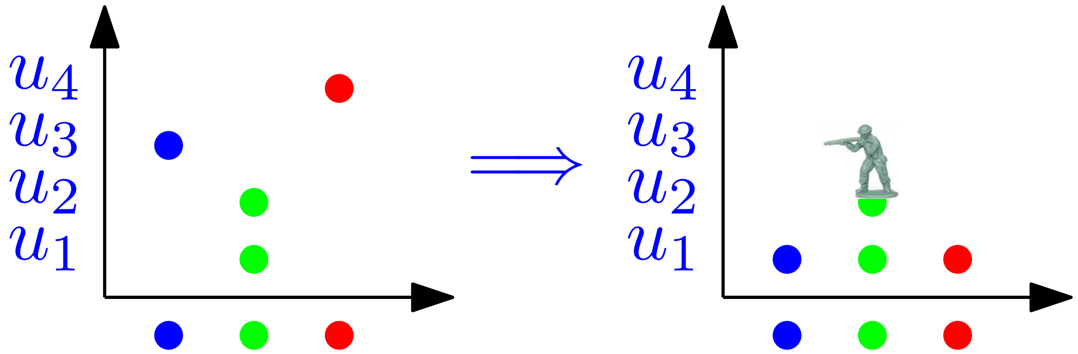

Coins are accumulated on few nodes

- When (u, v) interact:

$$new\ coins(u) = coins(u) \cap coins(v)$$
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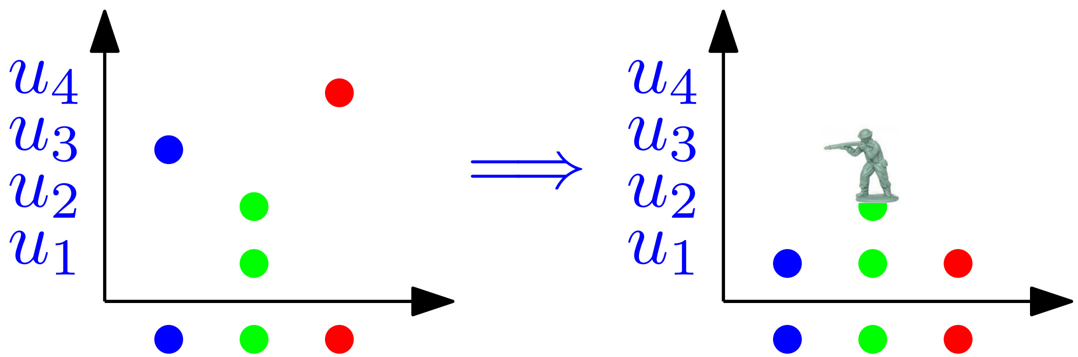

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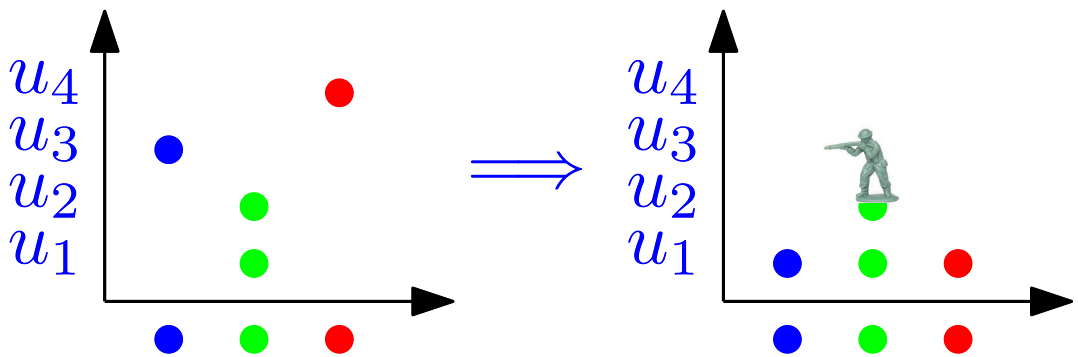

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Conjecture. $O(2^k)$ states are necessary.

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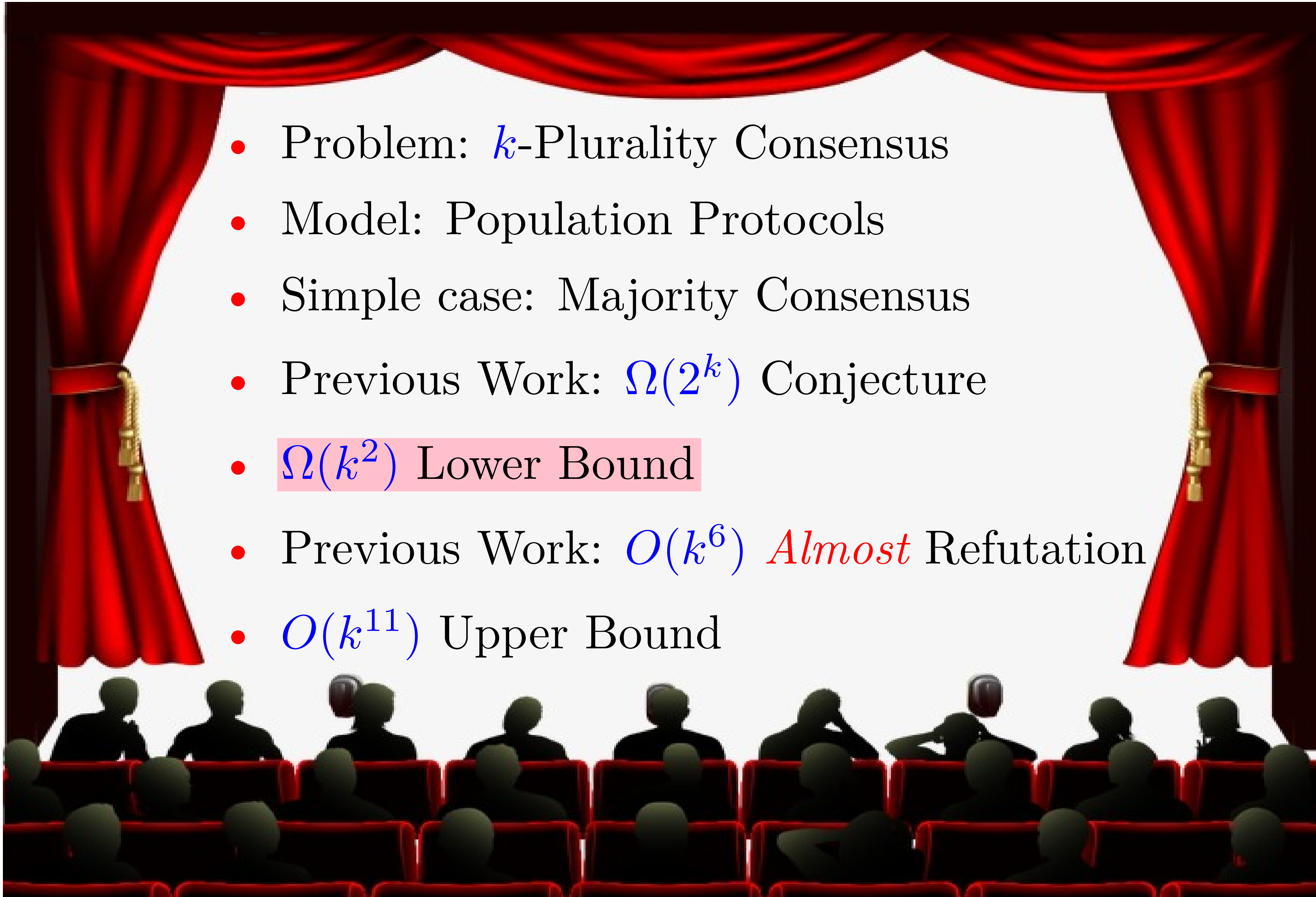

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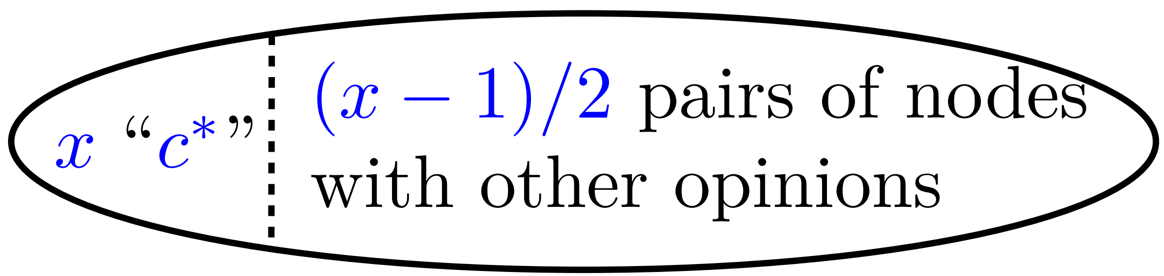

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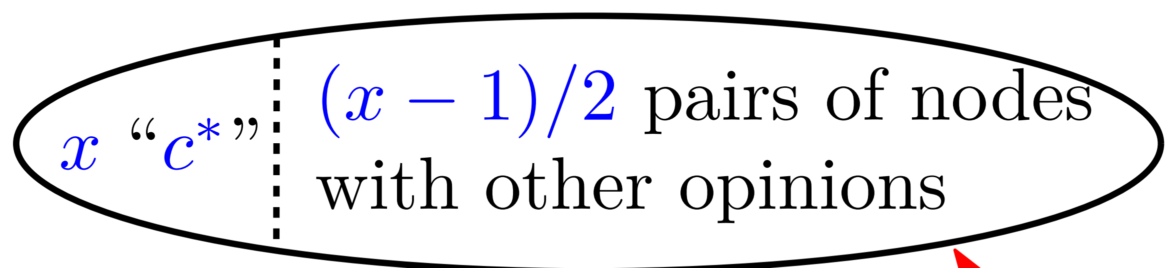
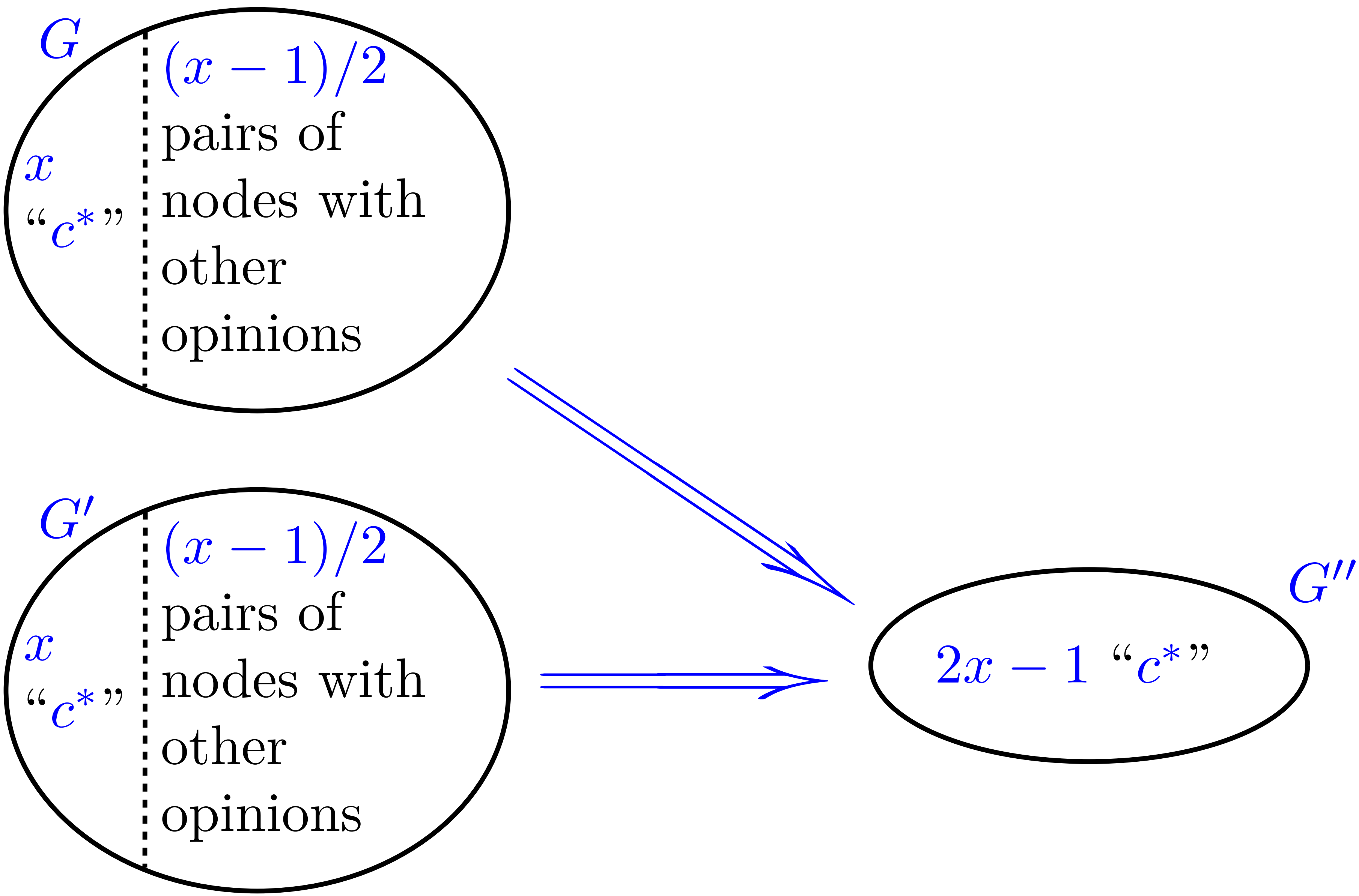


Pigeonhole: if $|\Sigma| < k^2 - k$, 2 config.s G and G' in •
converge to identical configurations.

$\Omega(k^2)$ Lower Bound II



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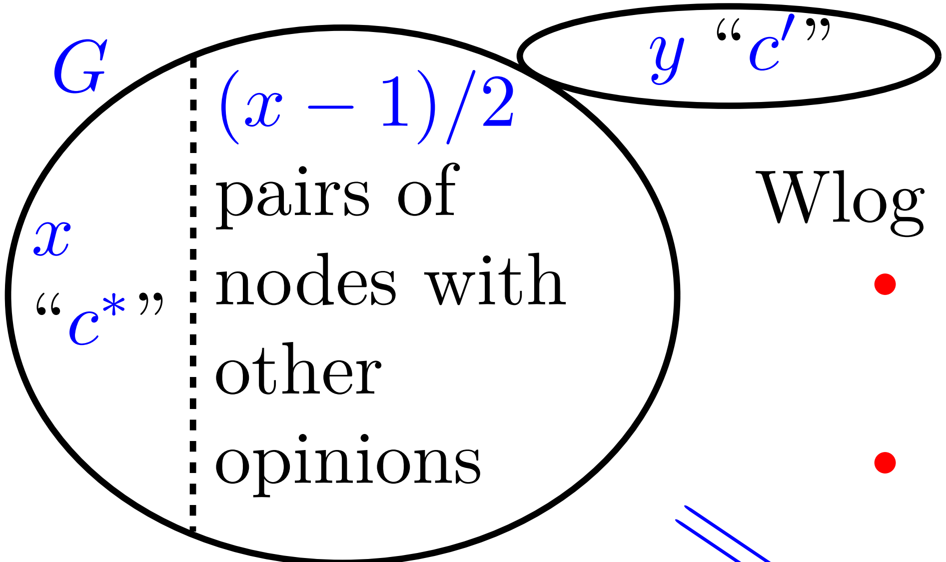
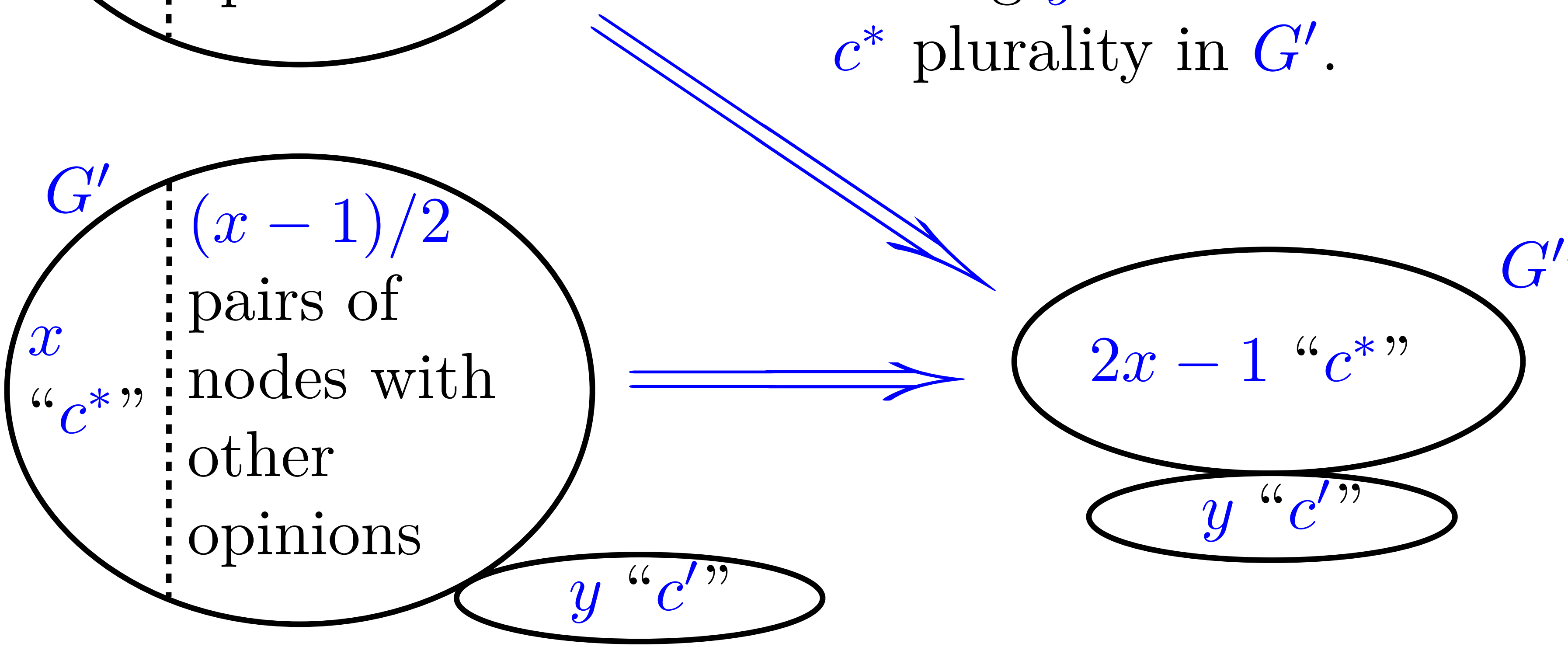


Wlog there is opinion c' and y s.t.

- adding y "c'" to G makes c' plurality in G
- adding y "c' to G' leaves c^* plurality in G'.



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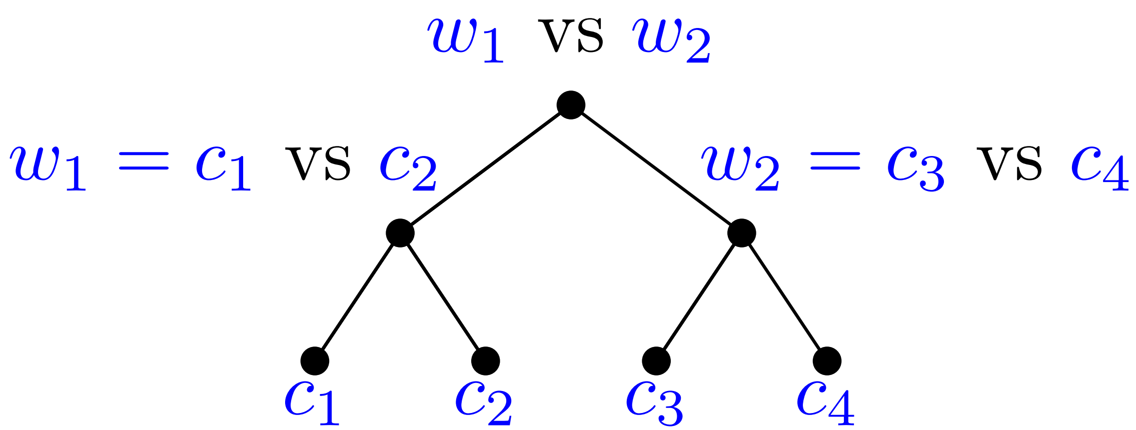
Plurality Consensus via Tournament Tree

Idea. Compute plurality by *majority* tournament.



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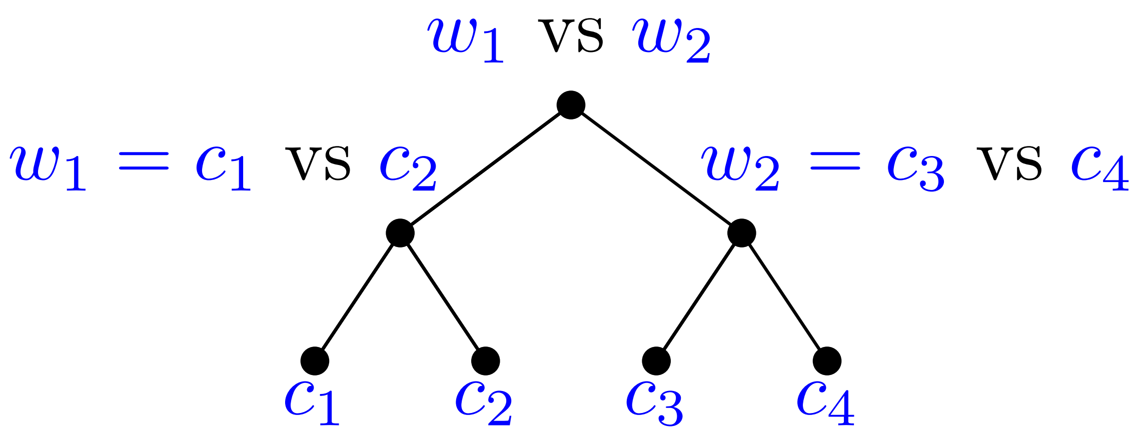

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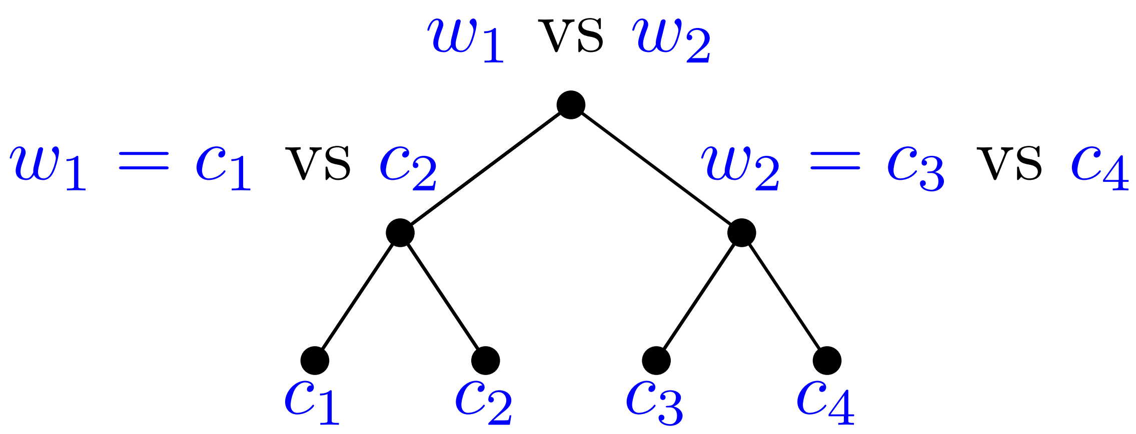

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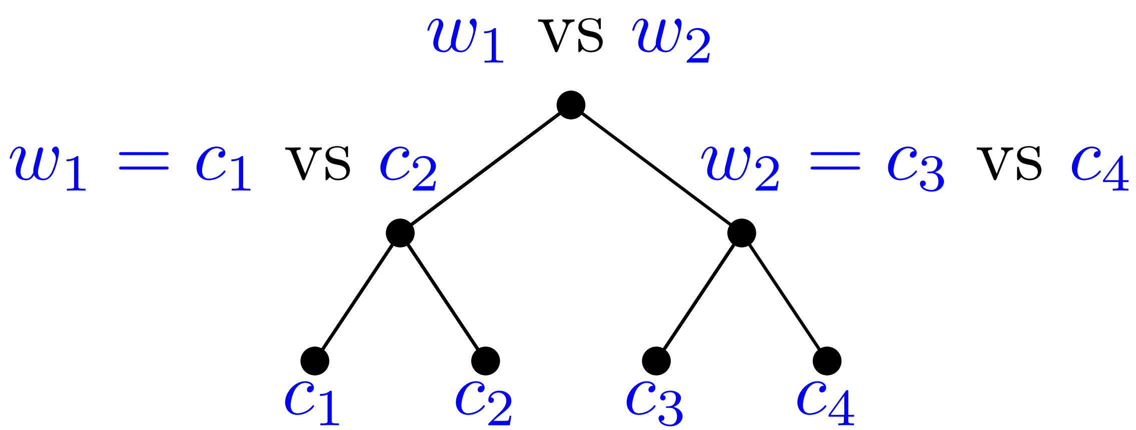

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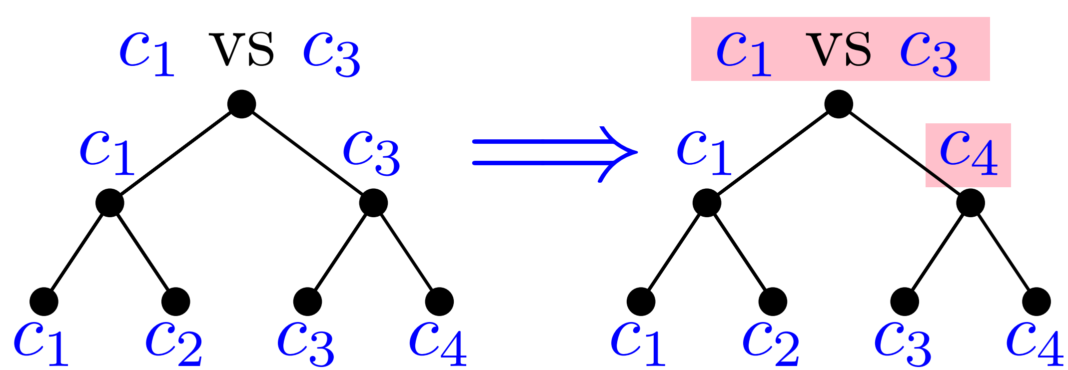

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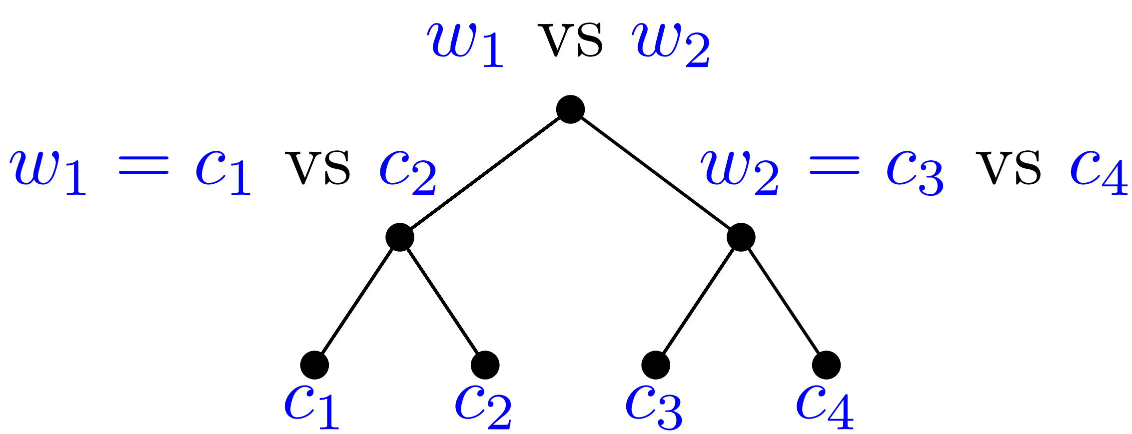

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Solved if nodes can change *opinion*.

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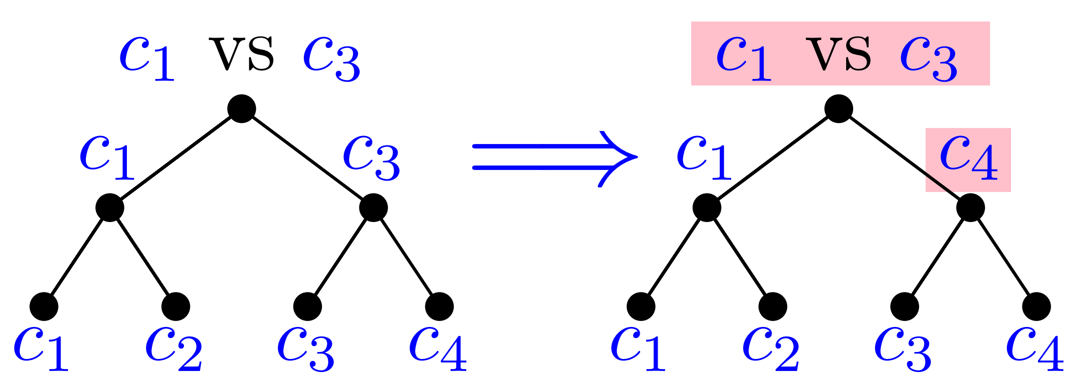

Dynamic Plurality Consensus

[Gasieniec et al. OPODIS'16]

Nodes can *change opinion* during execution.

States and weights

s	$w(s)$
$[-2]$	-2
$[-1]$	-1
$[0], \langle -1 \rangle, \langle 0 \rangle, \langle 1 \rangle$	0
$[1]$	1
$[2]$	2

Updating the state

s_a, $c_a = 1$ changes to $c'_a = -1$	s'_a
$[0], \langle -1 \rangle, \langle 0 \rangle, \langle 1 \rangle$	$[-2]$
$[1]$	$[-1]$
$[2]$	$[0]$
s_a, $c_a = -1$ changes to $c'_a = 1$	s'_a
$[0], \langle -1 \rangle, \langle 0 \rangle, \langle 1 \rangle$	$[2]$
$[-1]$	$[1]$
$[-2]$	$[0]$

Transitions

$s_a \backslash s_b$	$[-2]$	$[-1]$	$[0]$	$[1]$	$[2]$
$[-2]$	$([-2], [-2])$	$([-2], [-1])$	$([-2], \langle -1 \rangle)$	$([-1], \langle -1 \rangle)$	$([0], [0])$
$[-1]$	$([-1], [-2])$	$([-1], [-1])$	$([-1], \langle -1 \rangle)$	$([0], [0])$	$(\langle 1 \rangle, [1])$
$[0]$	$(\langle -1 \rangle, [-2])$	$(\langle -1 \rangle, [-1])$	$([0], [0])$	$(\langle 1 \rangle, [1])$	$(\langle 1 \rangle, [2])$
$[1]$	$(\langle -1 \rangle, [-1])$	$([0], [0])$	$(\langle 1 \rangle, [1])$	$([1], [1])$	$([1], [2])$
$[2]$	$([0], [0])$	$([1], \langle 1 \rangle)$	$([2], \langle 1 \rangle)$	$([2], [1])$	$([2], [2])$
weak	$(\langle -1 \rangle, [-2])$	$(\langle -1 \rangle, [-1])$	$(\langle 0 \rangle, [0])$	$(\langle 1 \rangle, [1])$	$(\langle 1 \rangle, [2])$

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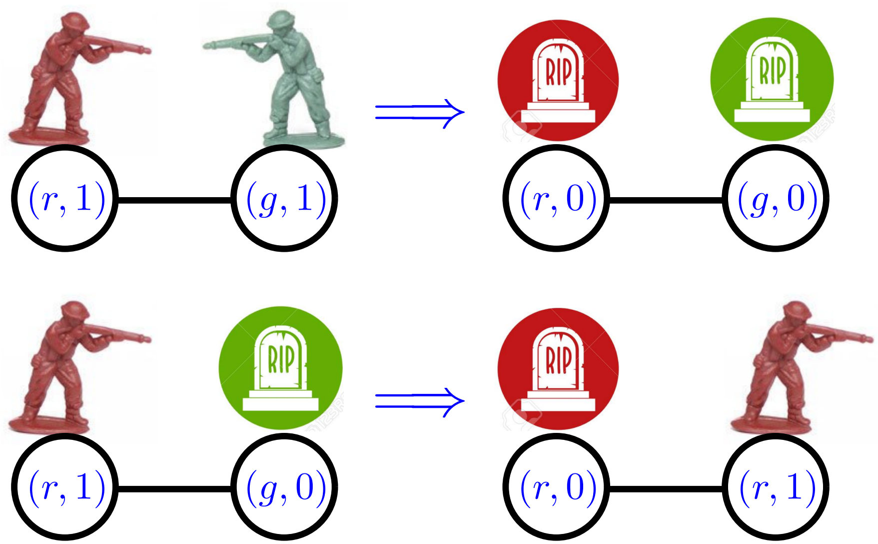

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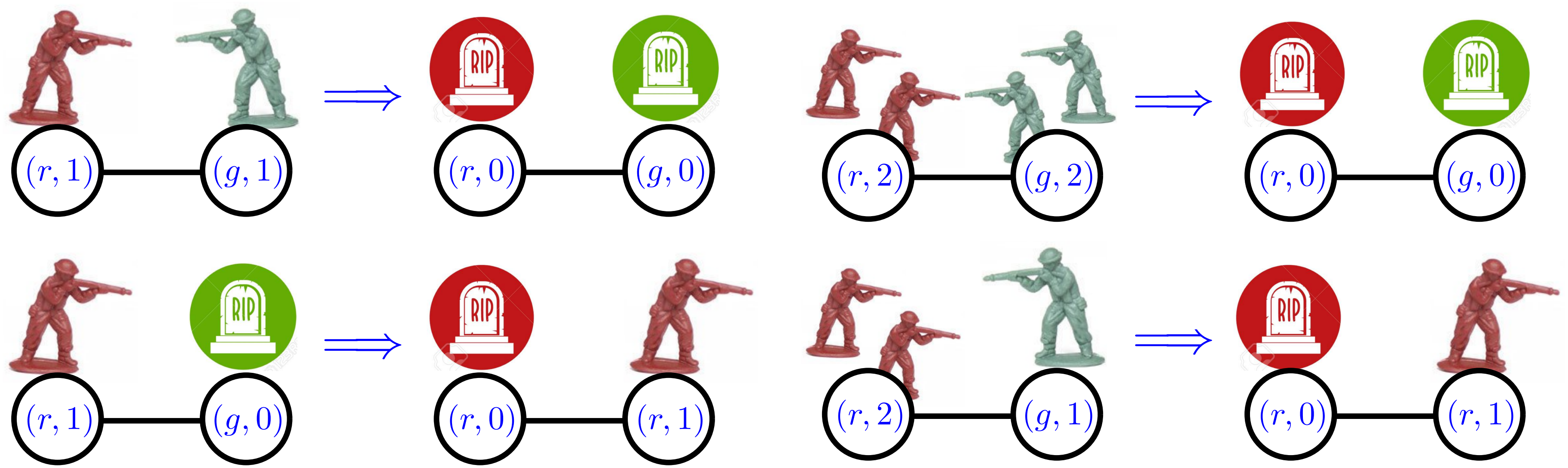

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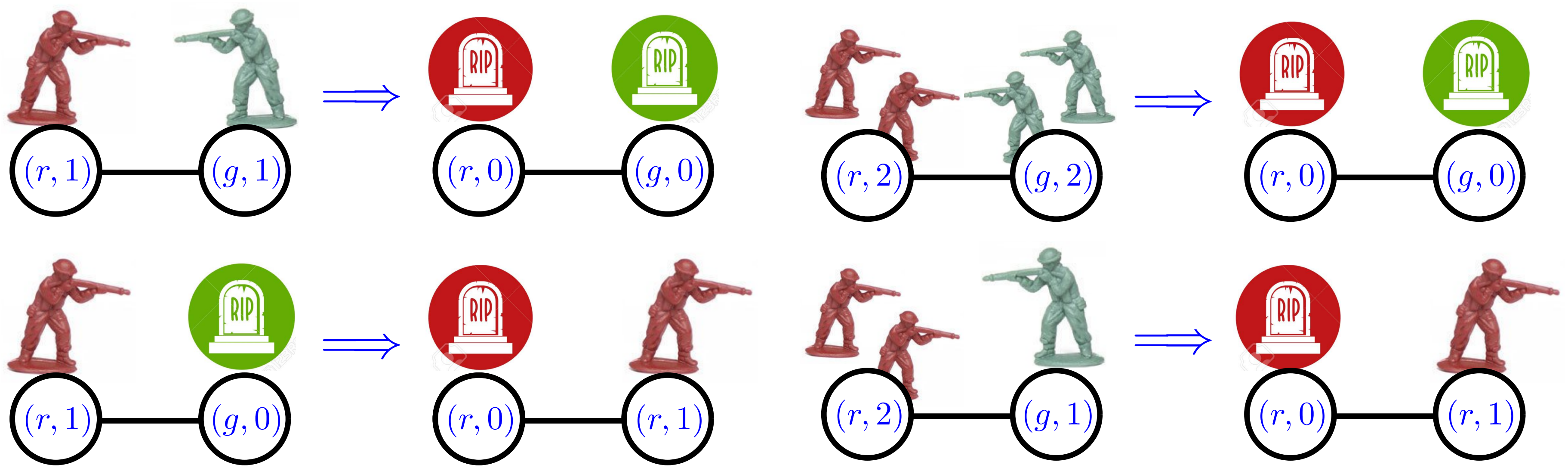

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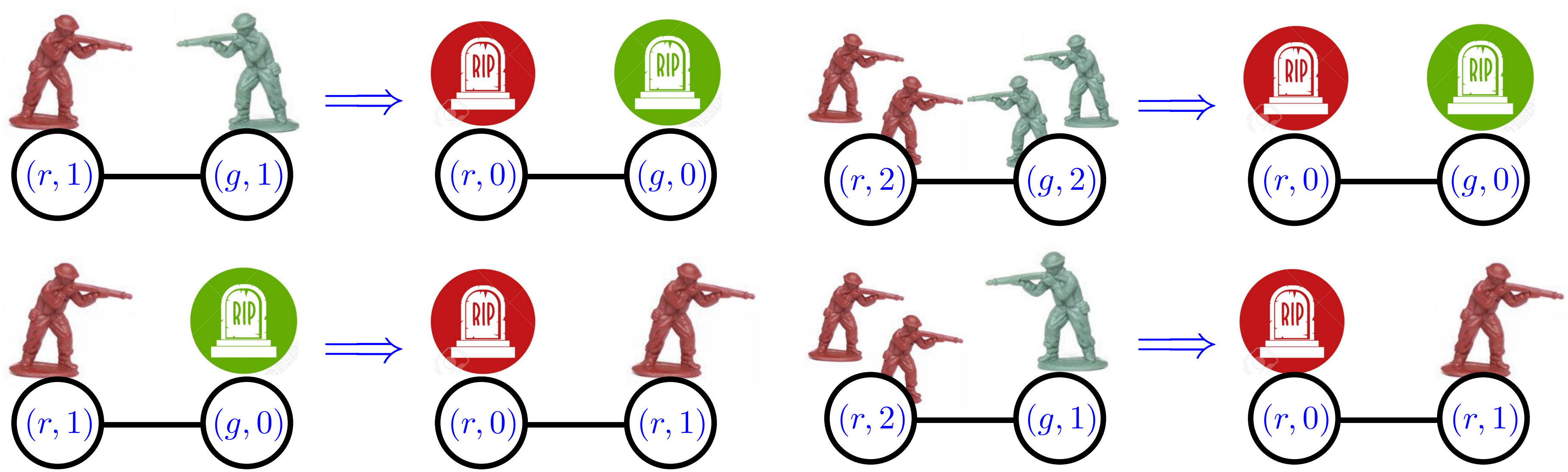

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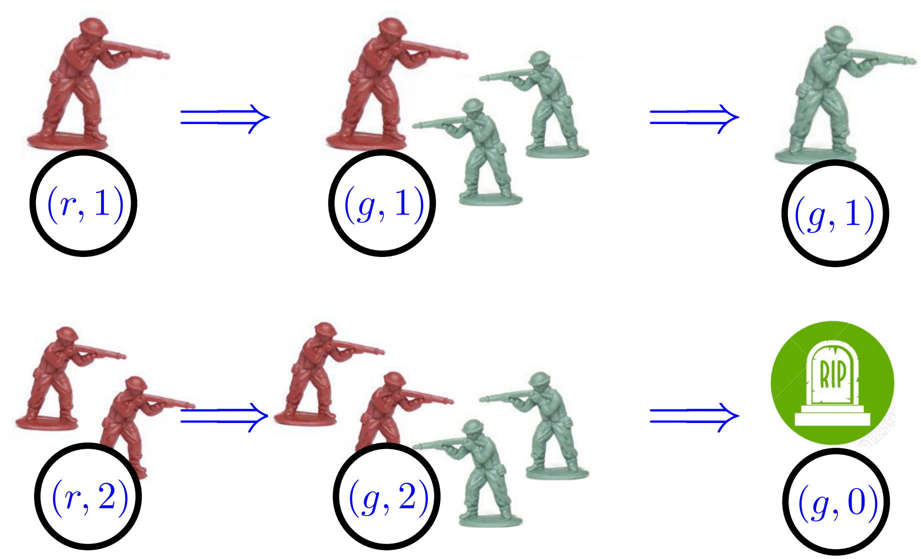

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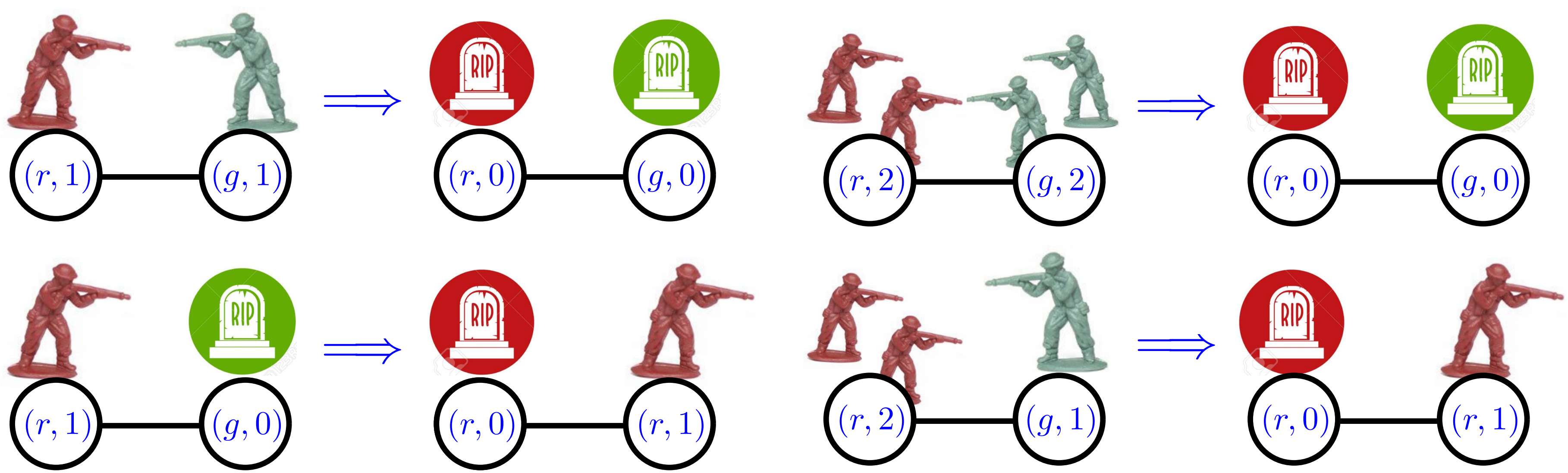

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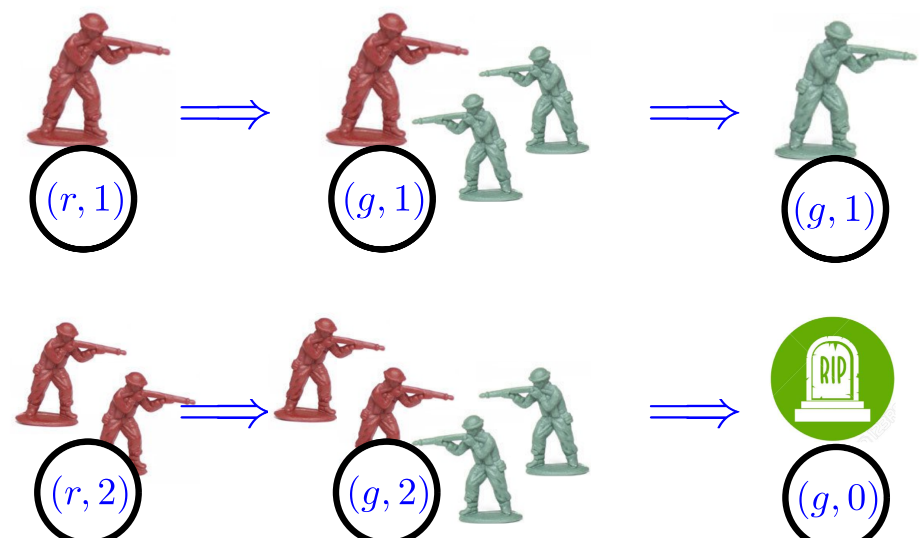
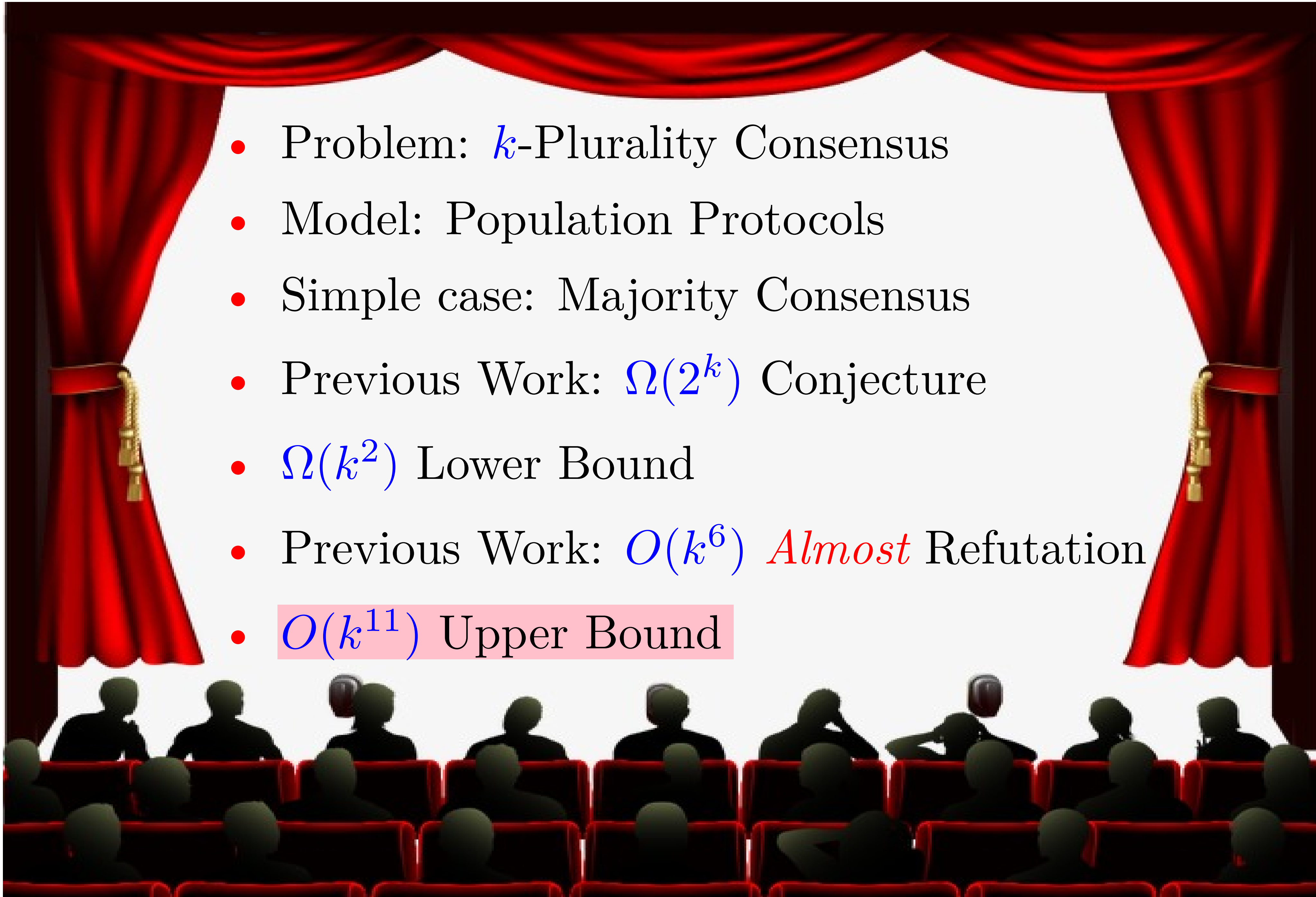


Balance of opinions

equals

balance of soldiers

Outline



- Problem: k-Plurality Consensus
- Model: Population Protocols
- Simple case: Majority Consensus
- Previous Work: $\Omega(2^k)$ Conjecture
- $\Omega(k^2)$ Lower Bound
- Previous Work: $O(k^6)$ *Almost* Refutation
- $O(k^{11})$ Upper Bound

$O(k^{11})$ Upper Bound (Refunting Conjecture)

To refute Salehkeleybar's conjecture we provide a protocol that *creates a labeling* and can run *in parallel* with Gasieniec et al.'s.

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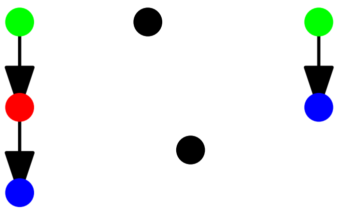
Idea. Have agents arrange opinions in a linked list.

Problem. Multiple lists can appears. How to delete/merge lists?



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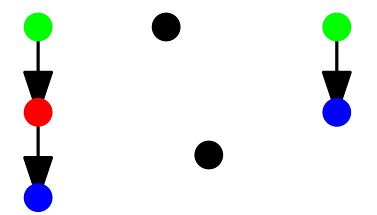
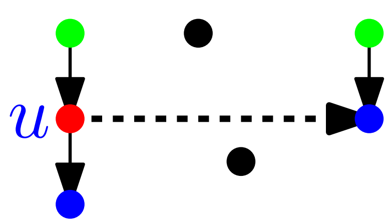
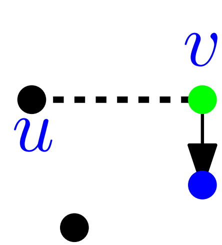


Ideas. Start deleting from *roots* of lists and append elements by travelling from root to last item.



u will inform parent that list shall be deleted.



u starts by designating v as parent. Eventually u designates as parents v's child, and so on.

Open Problem

Non-ordered self-stabilizing plurality consensus in population protocols with fair scheduler can be solved using $O(k^{11})$ states per agent.

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What is the space complexity of plurality consensus in population protocols with fair scheduler?

Thank You