Natural Distributed Algorithms

- Lecture 6 -
Necessary Memory for Majority in Population Protocols

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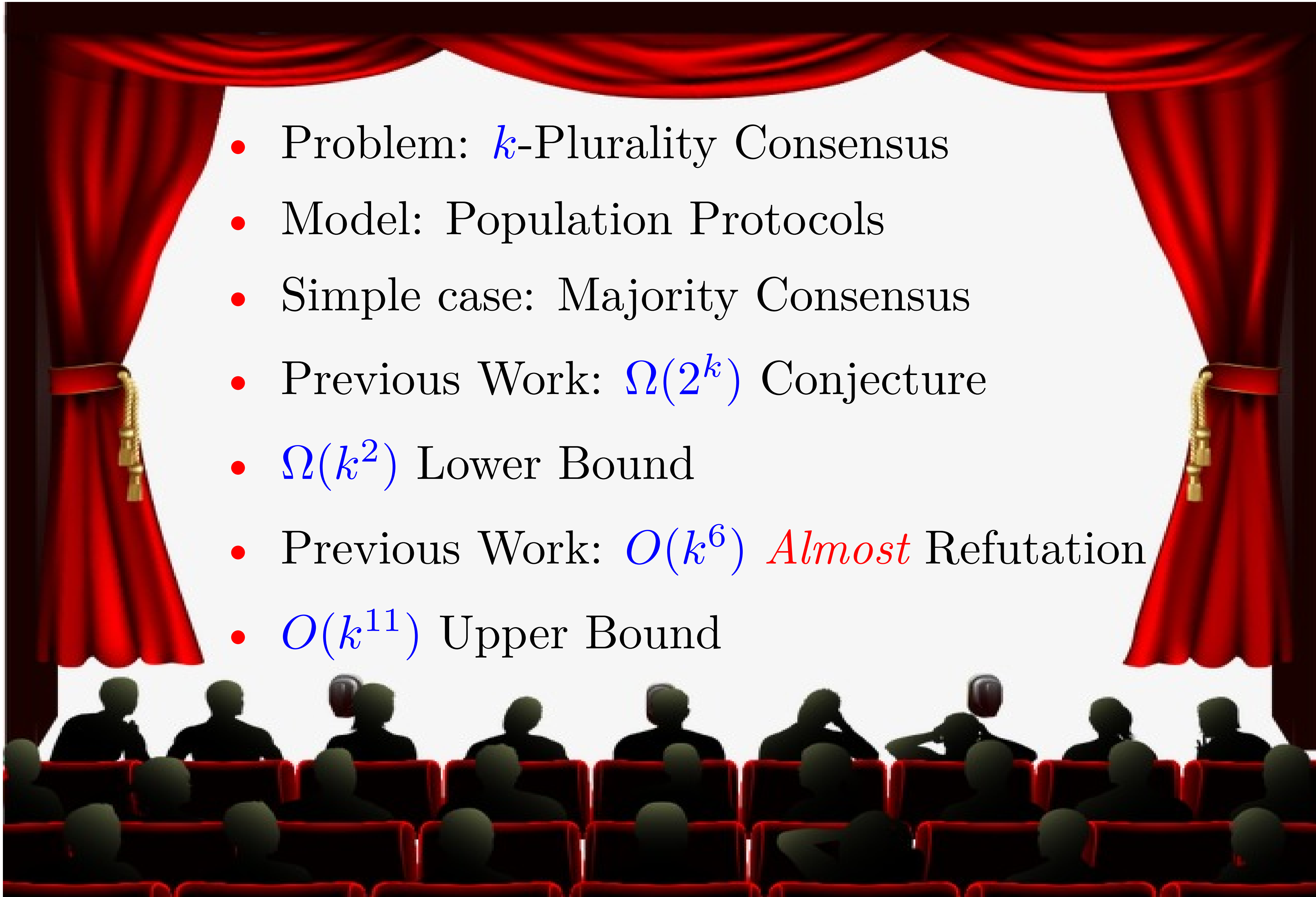
"Tor Vergata"

Outline



- Problem: k-Plurality Consensus
- Model: Population Protocols
- Simple case: Majority Consensus
- Previous Work: $\Omega(2^k)$ Conjecture
- $\Omega(k^2)$ Lower Bound
- Previous Work: $O(k^6)$ *Almost* Refutation
- $O(k^{11})$ Upper Bound

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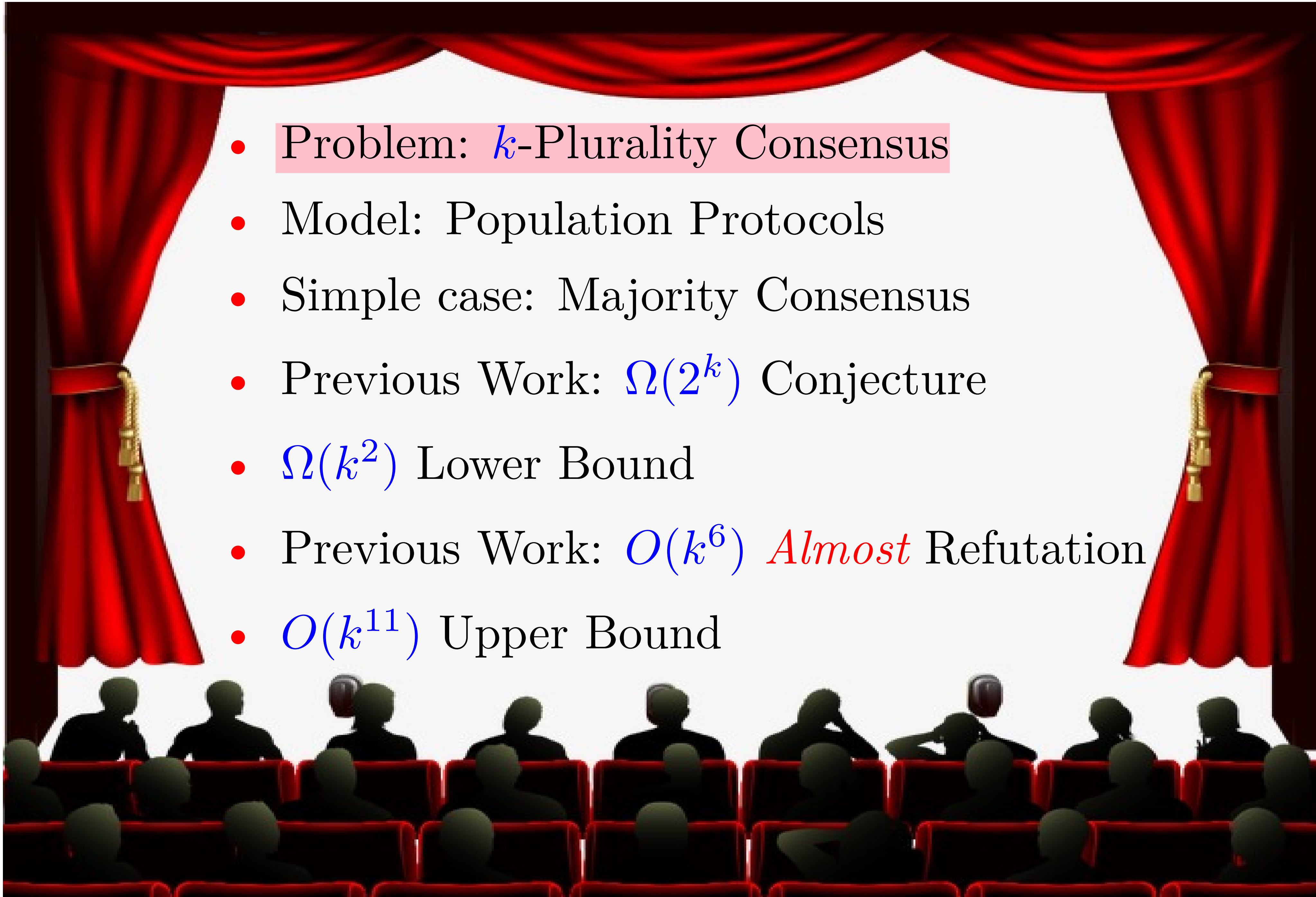

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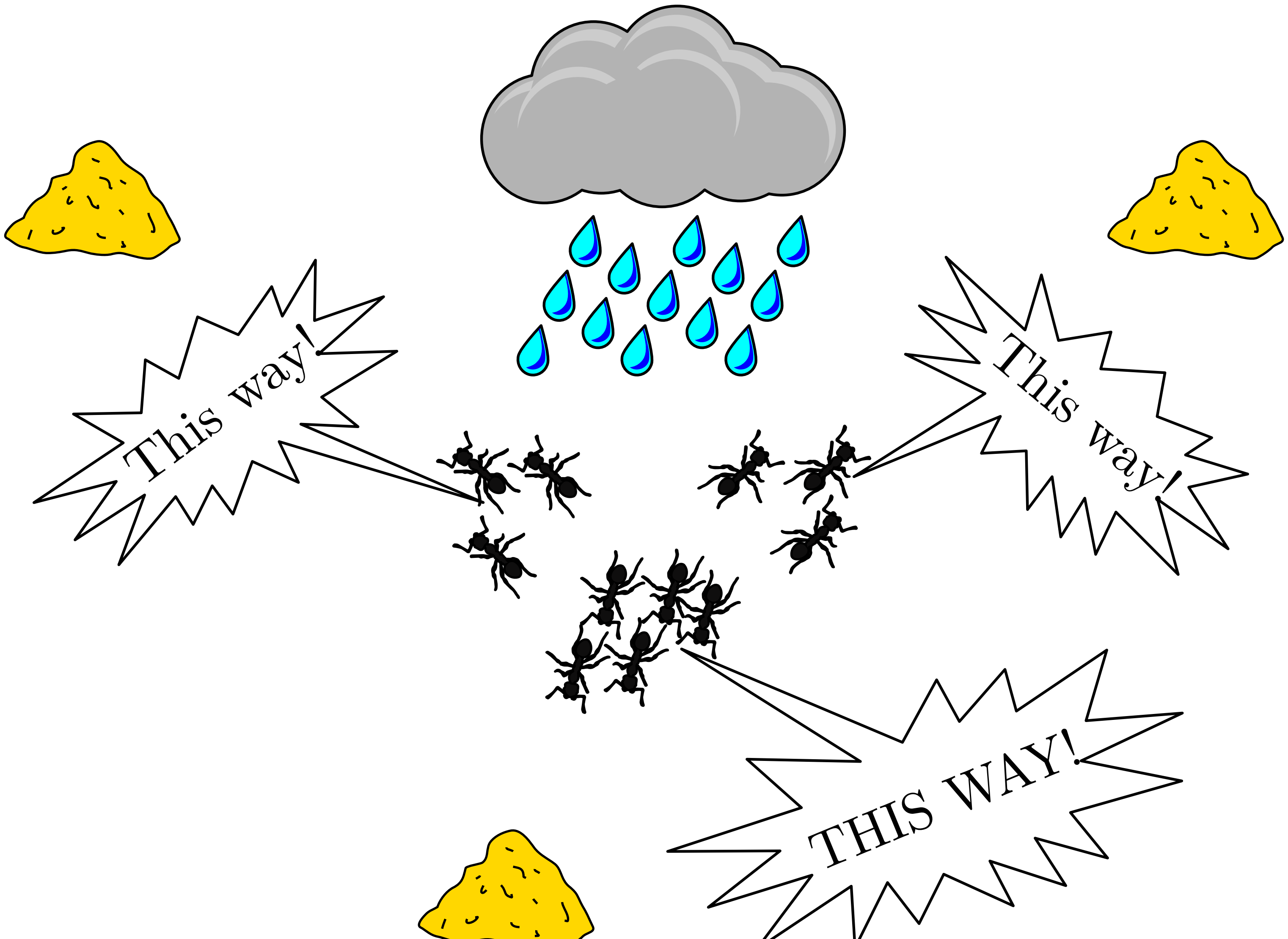
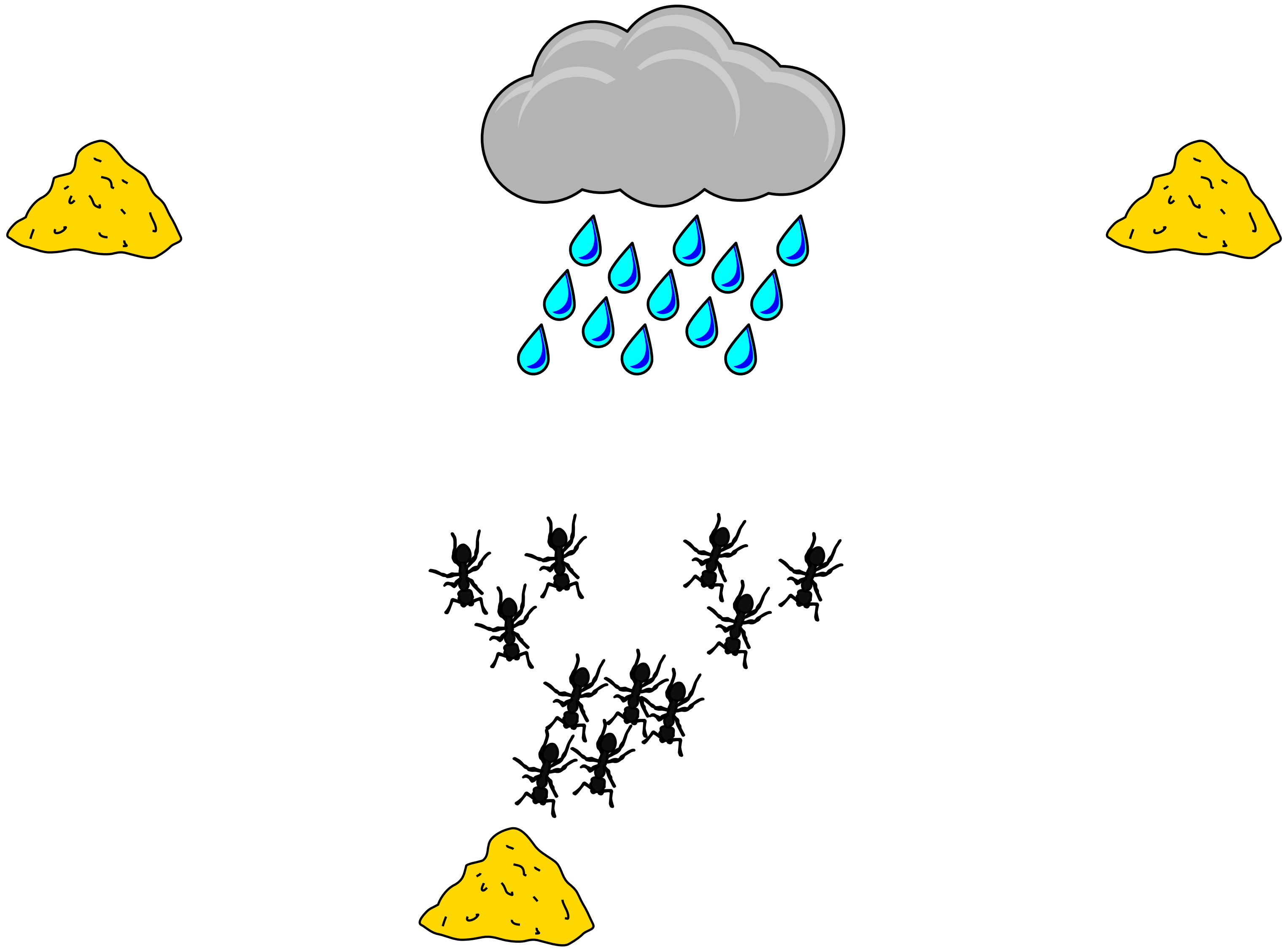
Recall: k-Plurality Consensus

Each agent supports one out of k opinions



Recall: k-Plurality Consensus

All agents eventually support the same opinion



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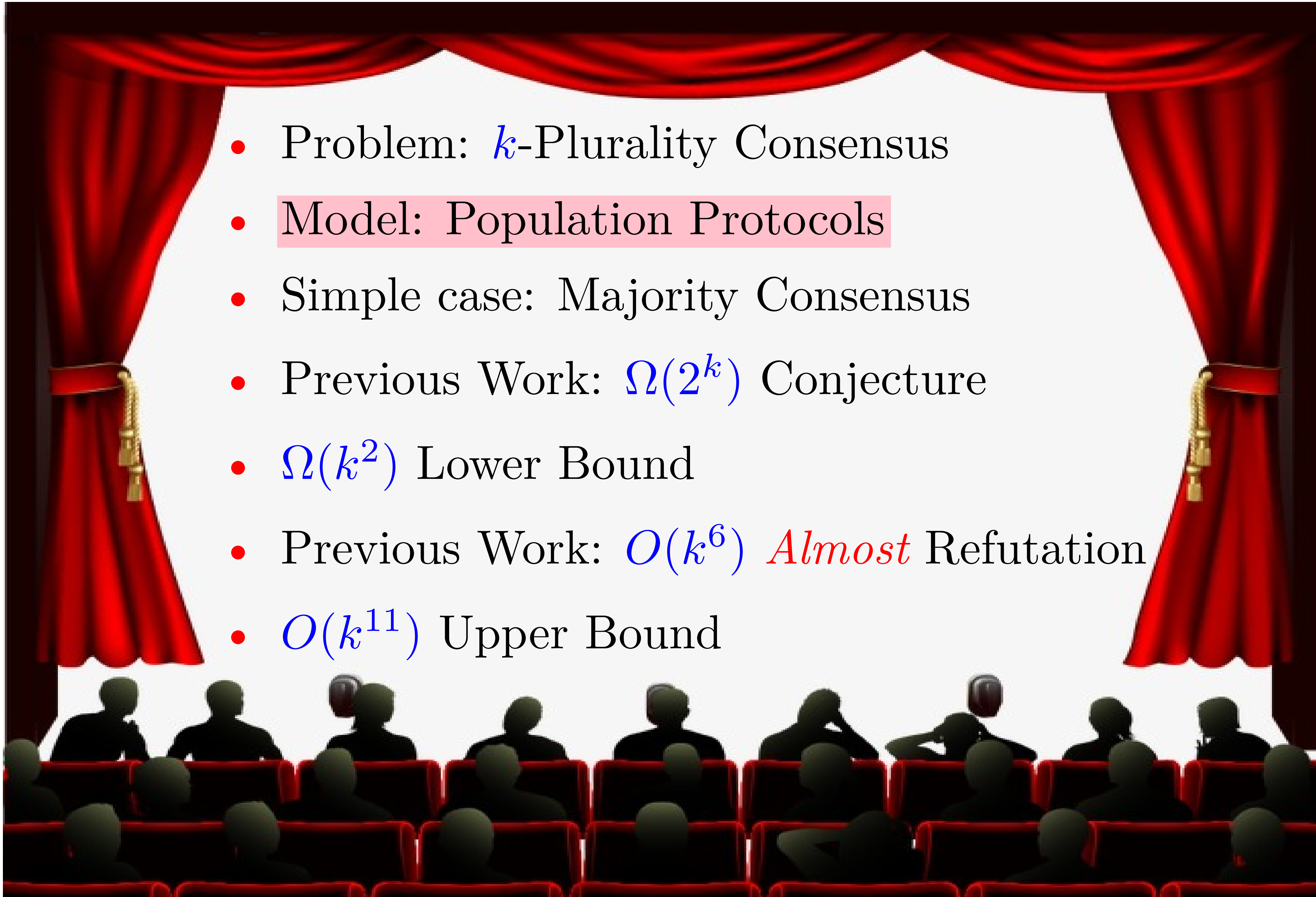

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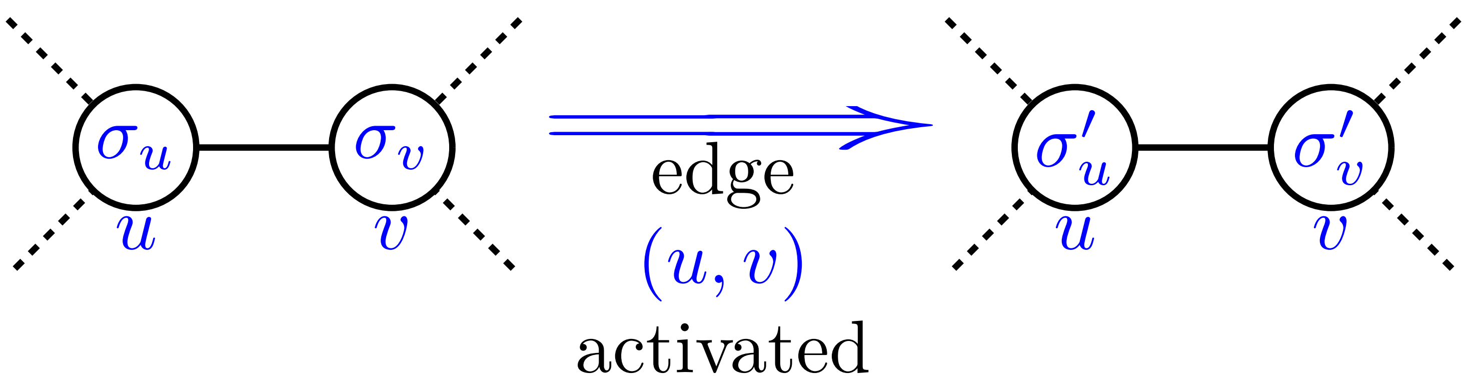
- (Directed) graph G,
- set of nodes' states $\Sigma = (\sigma_u)_{u \in V}$,
- edges activated by a *scheduler*,
- function $\gamma : \Sigma \times \Sigma \to \Sigma \times \Sigma$ s.t. if edge (u, v) with states (σ_u, σ_v) activated, new states are

$$\gamma(\sigma_u, \sigma_v) = (\sigma'_u, \sigma'_v)$$



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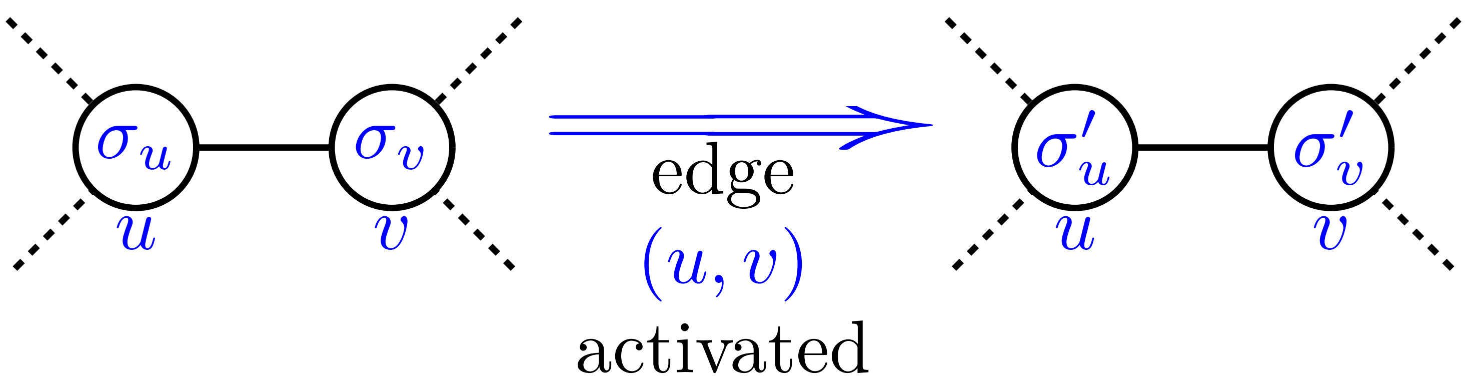
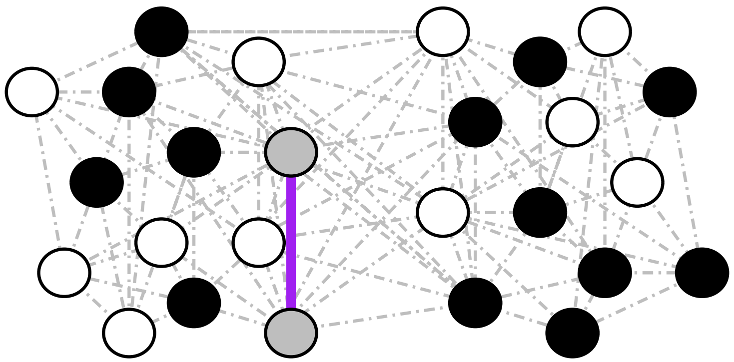
protocol's memory



Population Protocols: Schedulers

Probabilistic scheduler: activate an edge chosen at random



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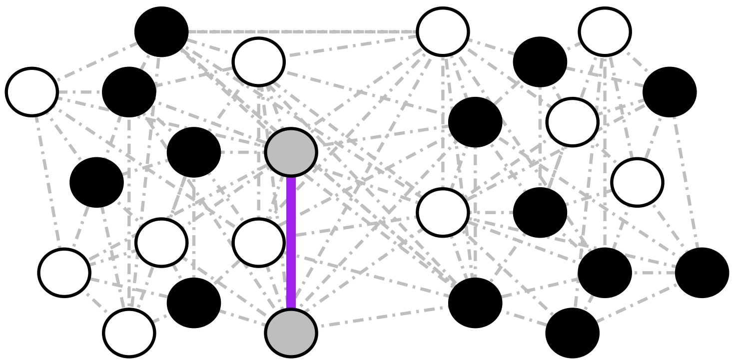
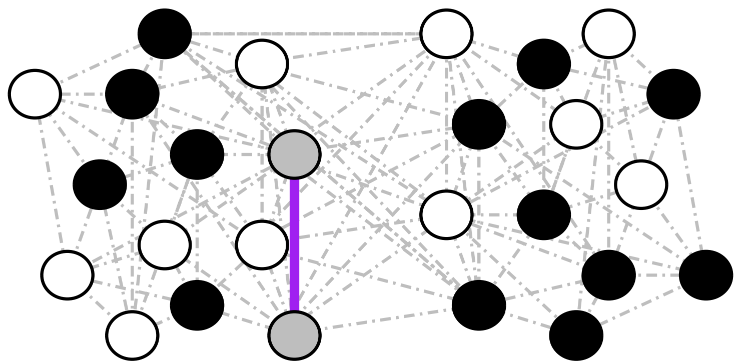


What if a protocol P should never fail?

Population Protocols: Schedulers



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A *configuration* is the state of all nodes $S = (\sigma_1, ..., \sigma_n)$.

S' *reachable* from S if it is possible to activate edges such that S becomes S'.

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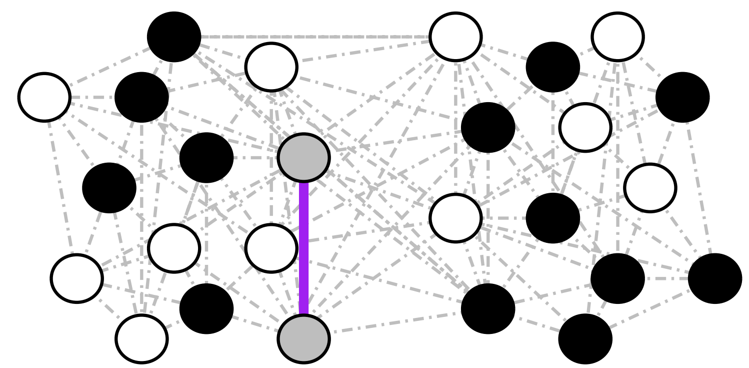

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Fair scheduler: if S appears infinitely often, also any conf. reachable from S appears infinitely often:

S' reachable from S and $S_1, S_2, ..., S, ..., S, ..., S, ...$

$\implies S_1, S_2, ..., S', ..., S', ..., S', ...$

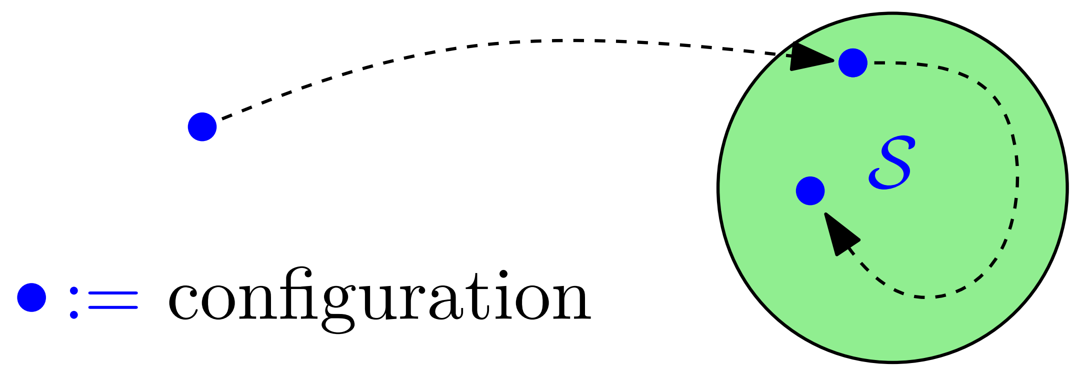
Self-Stabilization

n agents with states in Σ. Σ^n possible configurations.

$\mathcal{S}$:= {"*correct* states of the system" }.

Convergence. Starting from any possible configuration, the system eventually reaches a configuration in $\mathcal{S}$.
Closure. If configuration in $\mathcal{S}$, it remains in $\mathcal{S}$.

A protocol is self-stabilizing iff guarantees convergence and closure w.r.t. $\mathcal{S}$.



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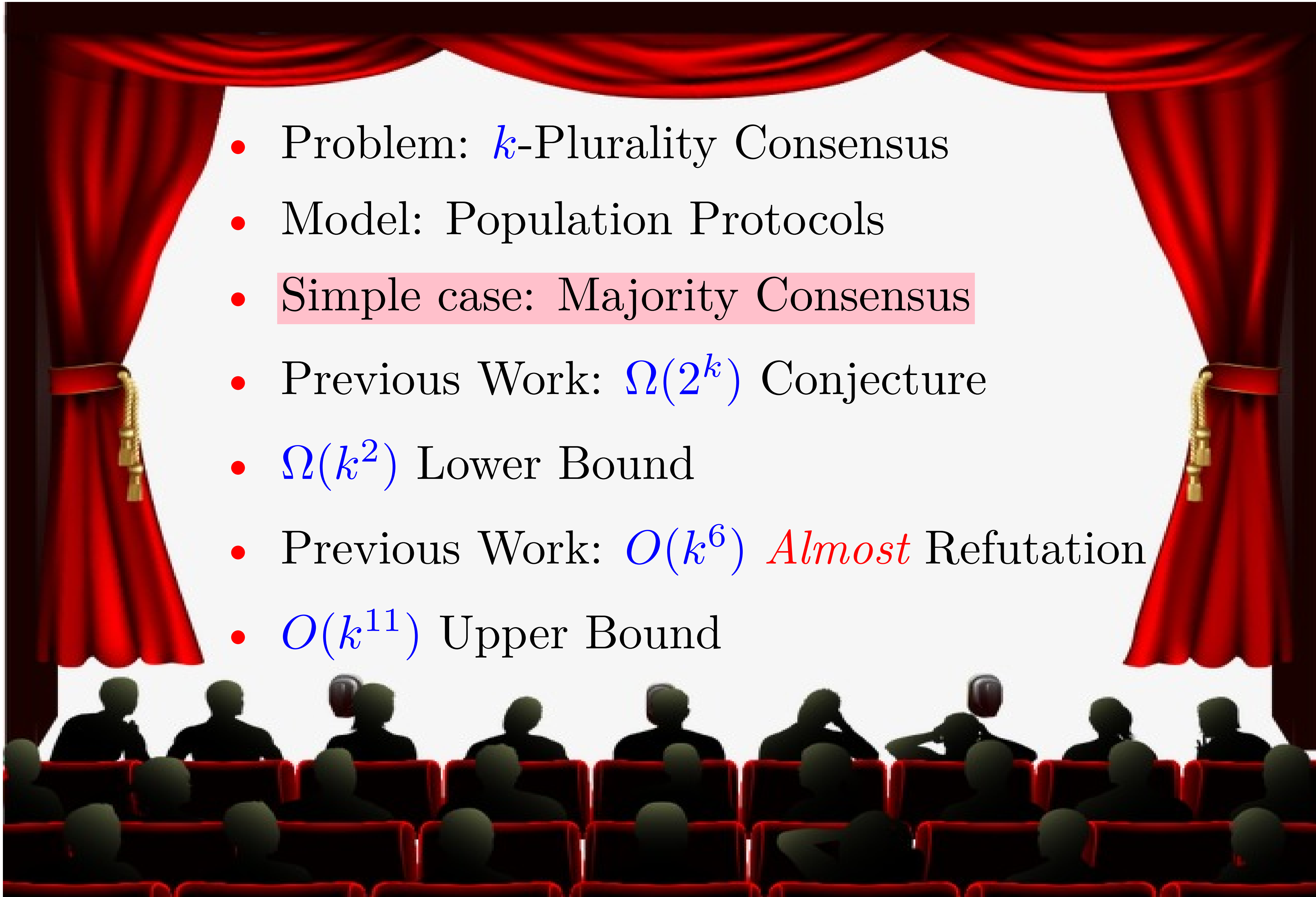

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Majority (2-Plurality) Consensus: 2-bit Protocol

[Mertzios et al. ICALP'16, Benezit et al. ICASSP'09]

State: (green/red, defended or not)

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$u \setminus v$	$(g,0)$	$(g,1)$	$(r,0)$	$(r,1)$
$(g,0)$	–	$((g,1),(g,0))$	–	$((r,1),(r,0))$
$(g,1)$	$((g,0),(g,1))$	–	$((g,0),(g,1))$	$((g,0),(r,0))$
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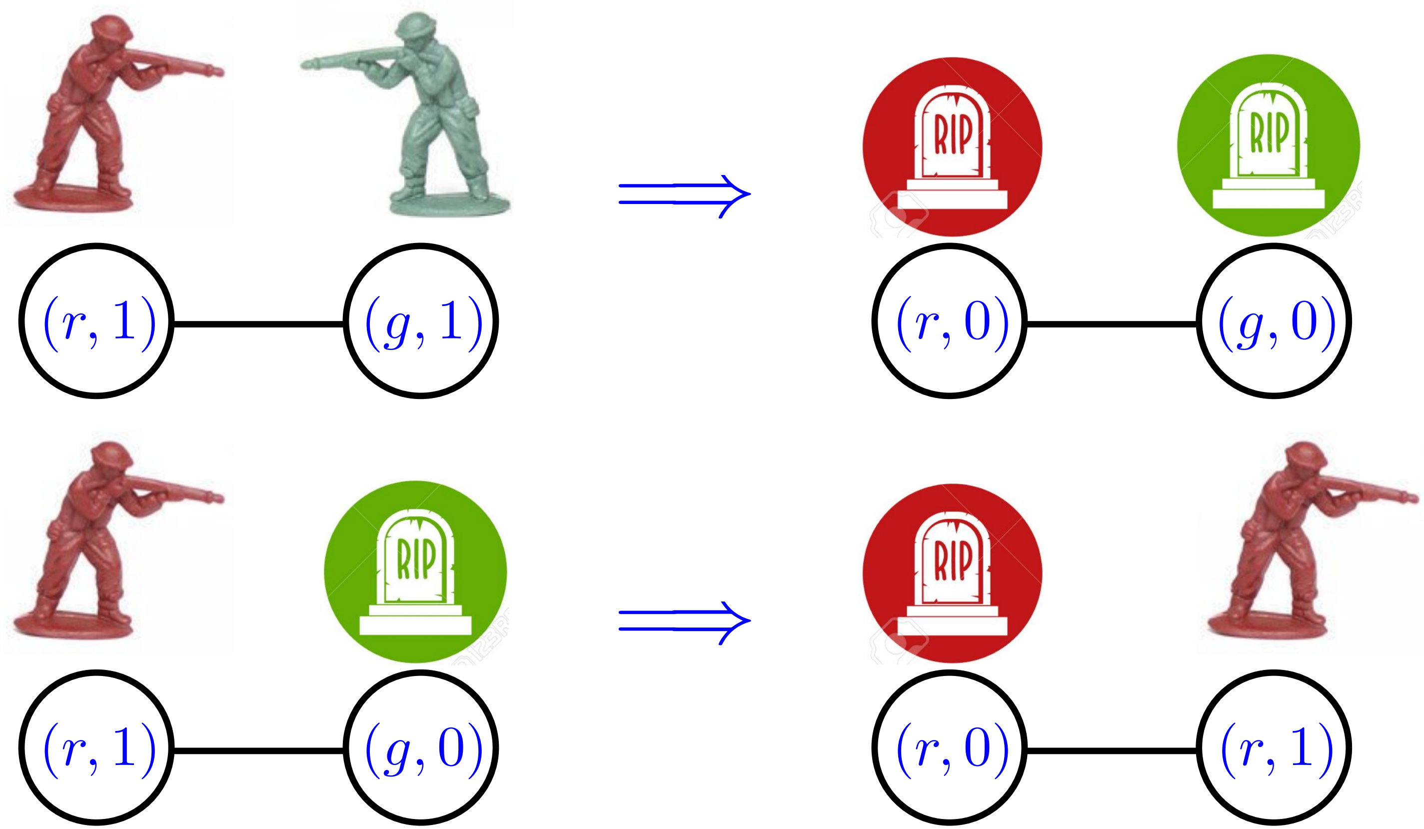

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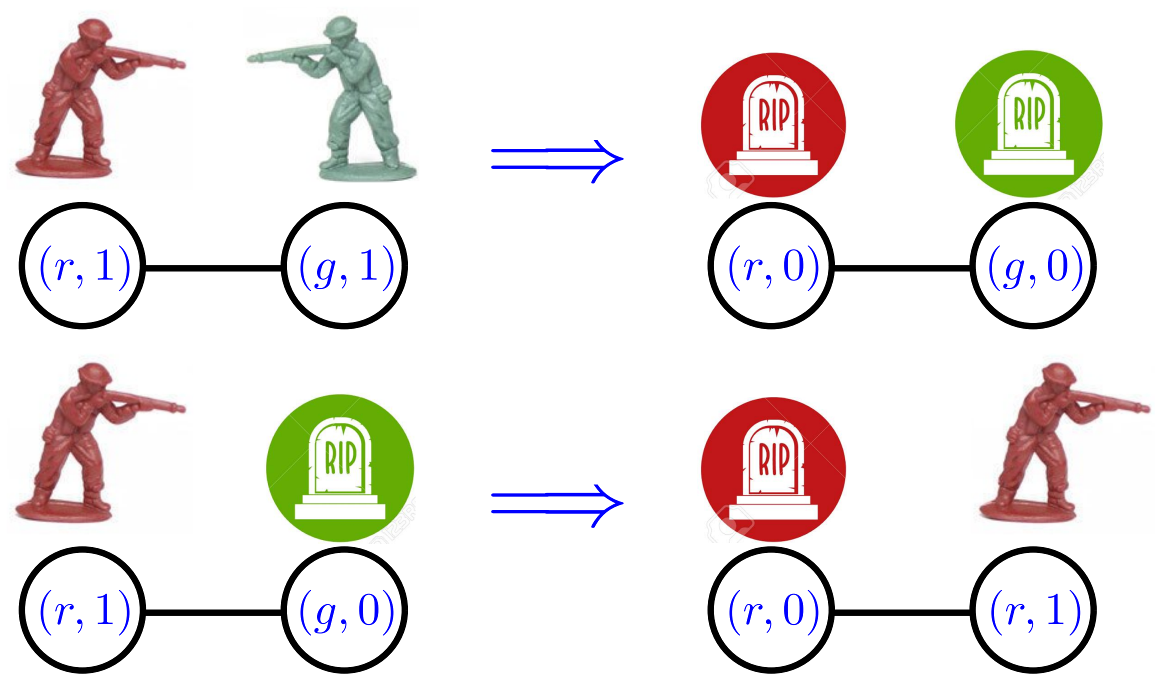

- If there is a clear majority, at some point there is only one type (green or red) of "strong agent"
- At some point the strong agent visits all nodes

Majority Consensus: 2-bit Lower Bound

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Three possible states: $1, 0, \alpha$.

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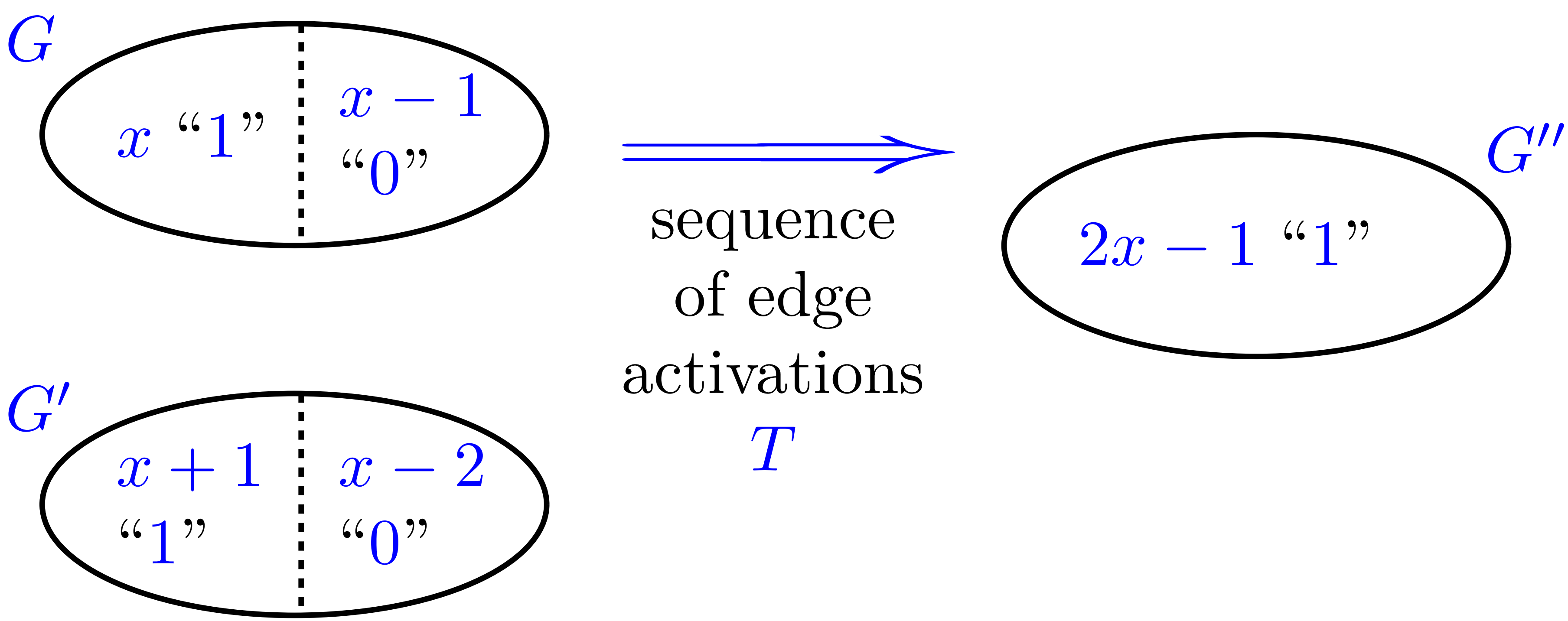

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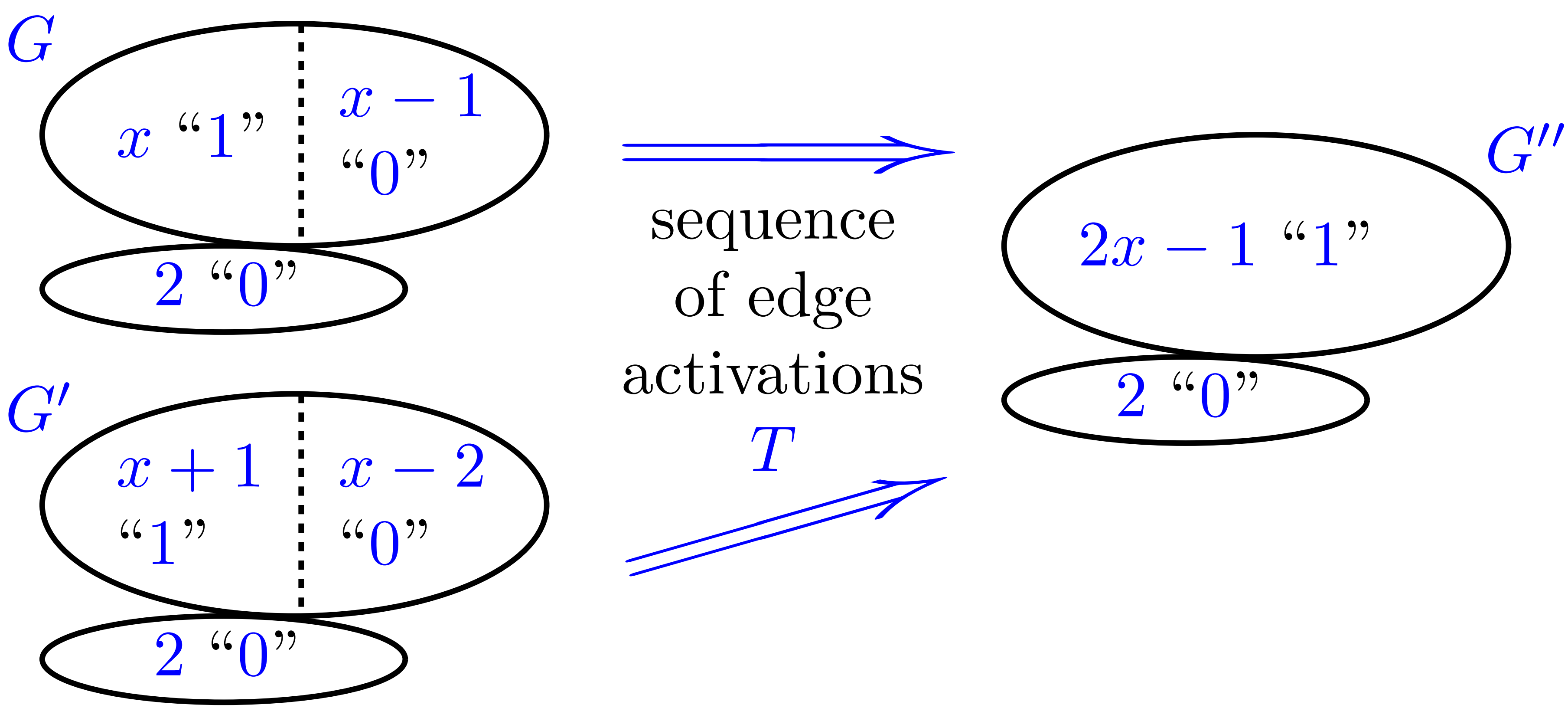

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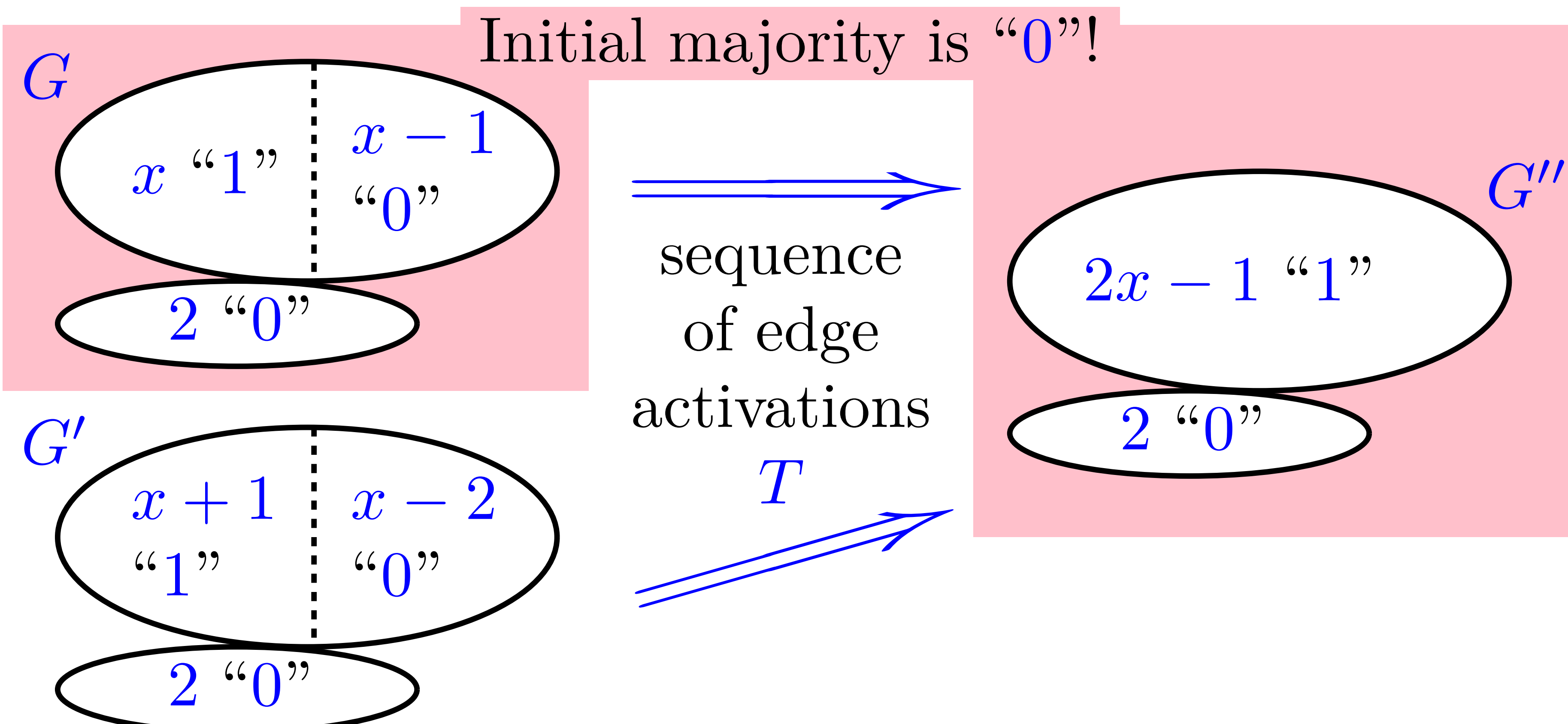

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Opinions can *only be tested for equality*.

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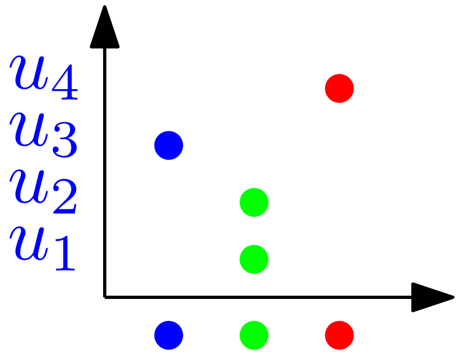

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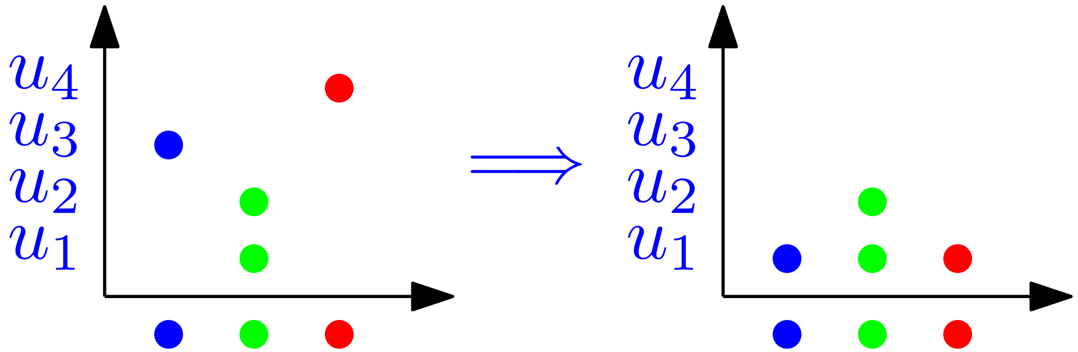

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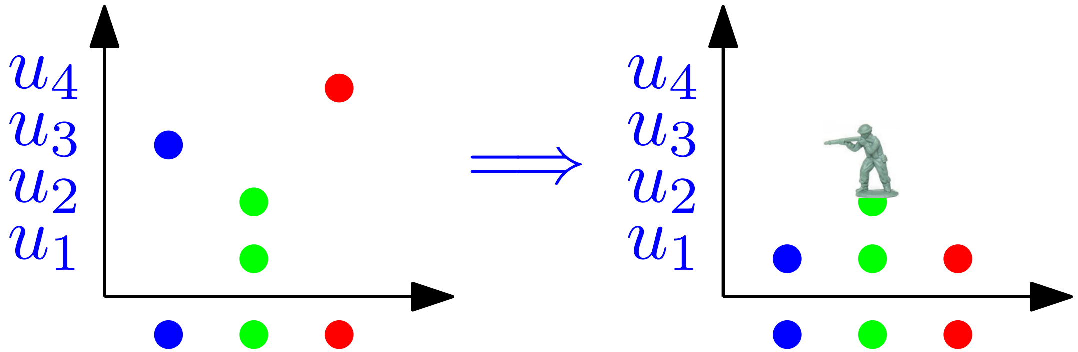

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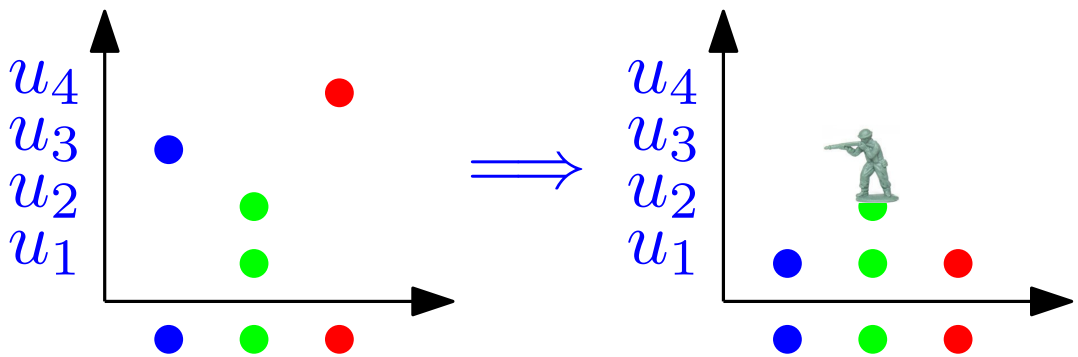

Coins are accumulated on few nodes

- When (u, v) interact:

$$\boxed{\begin{array}{l} new\ coins(u) = coins(u) \cap coins(v) \\ new\ coins(v) = coins(u) \cup coins(v) \end{array}}$$

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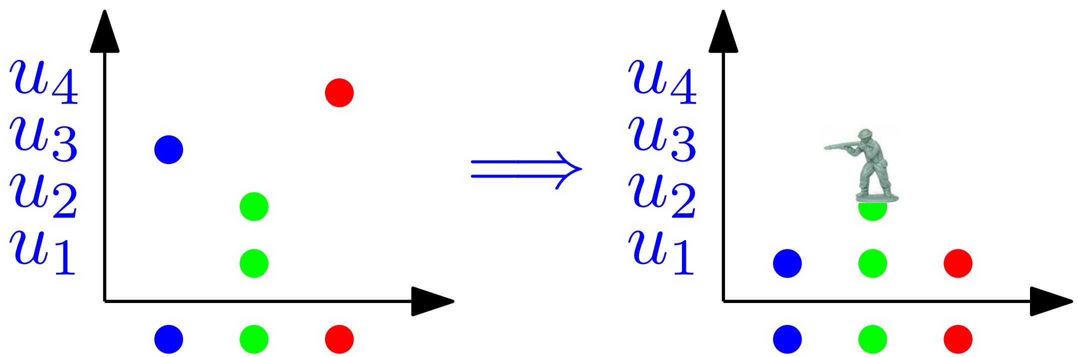

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$\sum_v |\text{coins}(v)|^2$

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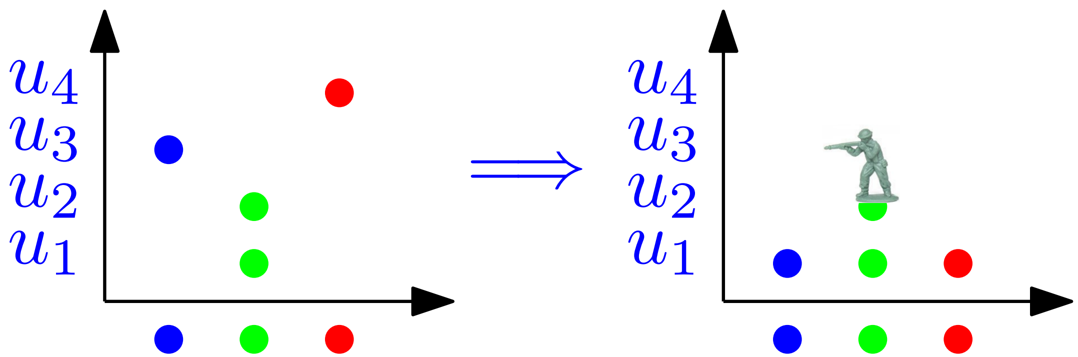

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Conjecture. $O(2^k)$ states are necessary.

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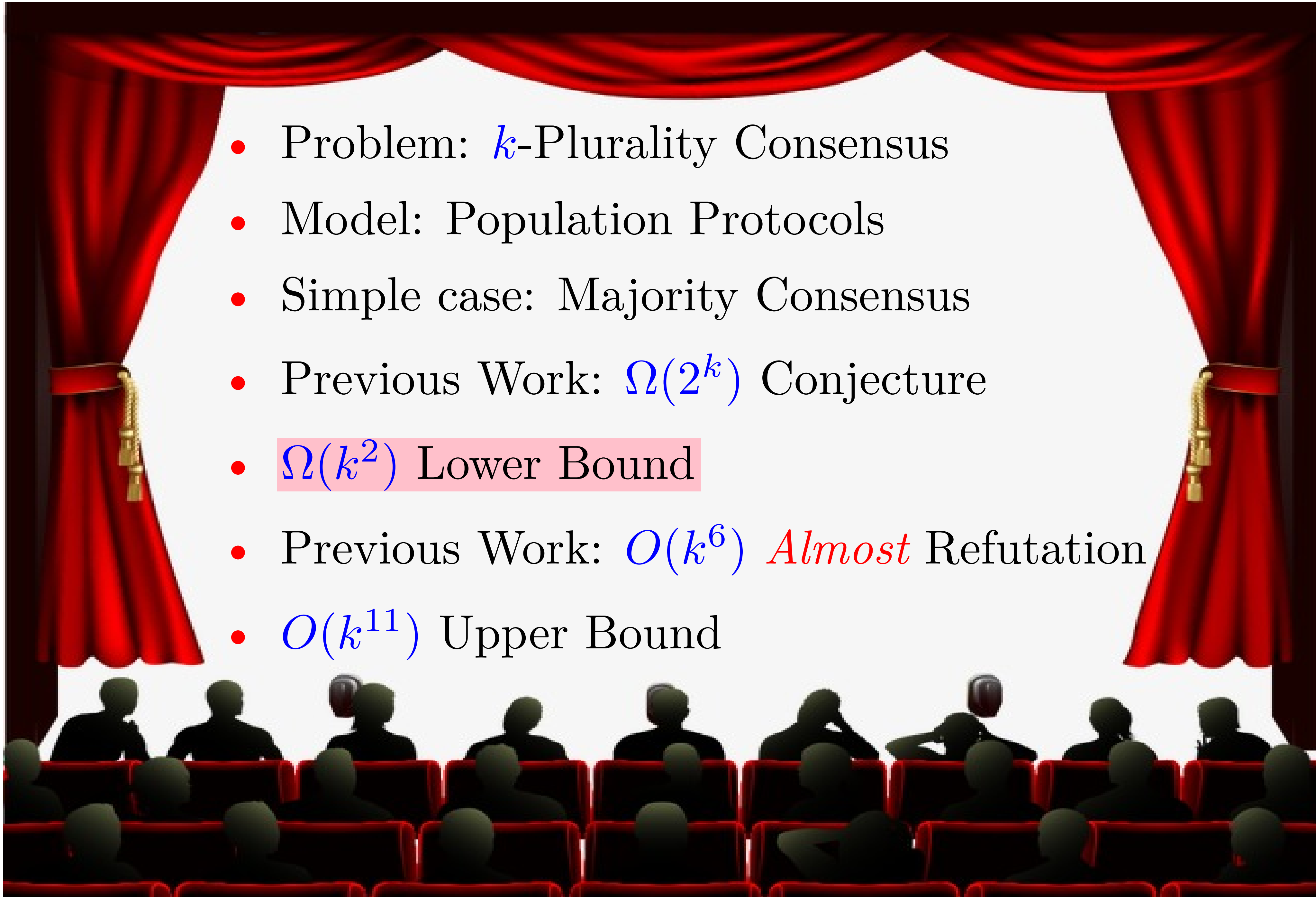

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In at most $\approx \left(\frac{2e \cdot x}{\frac{|\Sigma|}{k} - 1}\right)^{\frac{|\Sigma|}{k} - 1}$ config.s all nodes output c^*.

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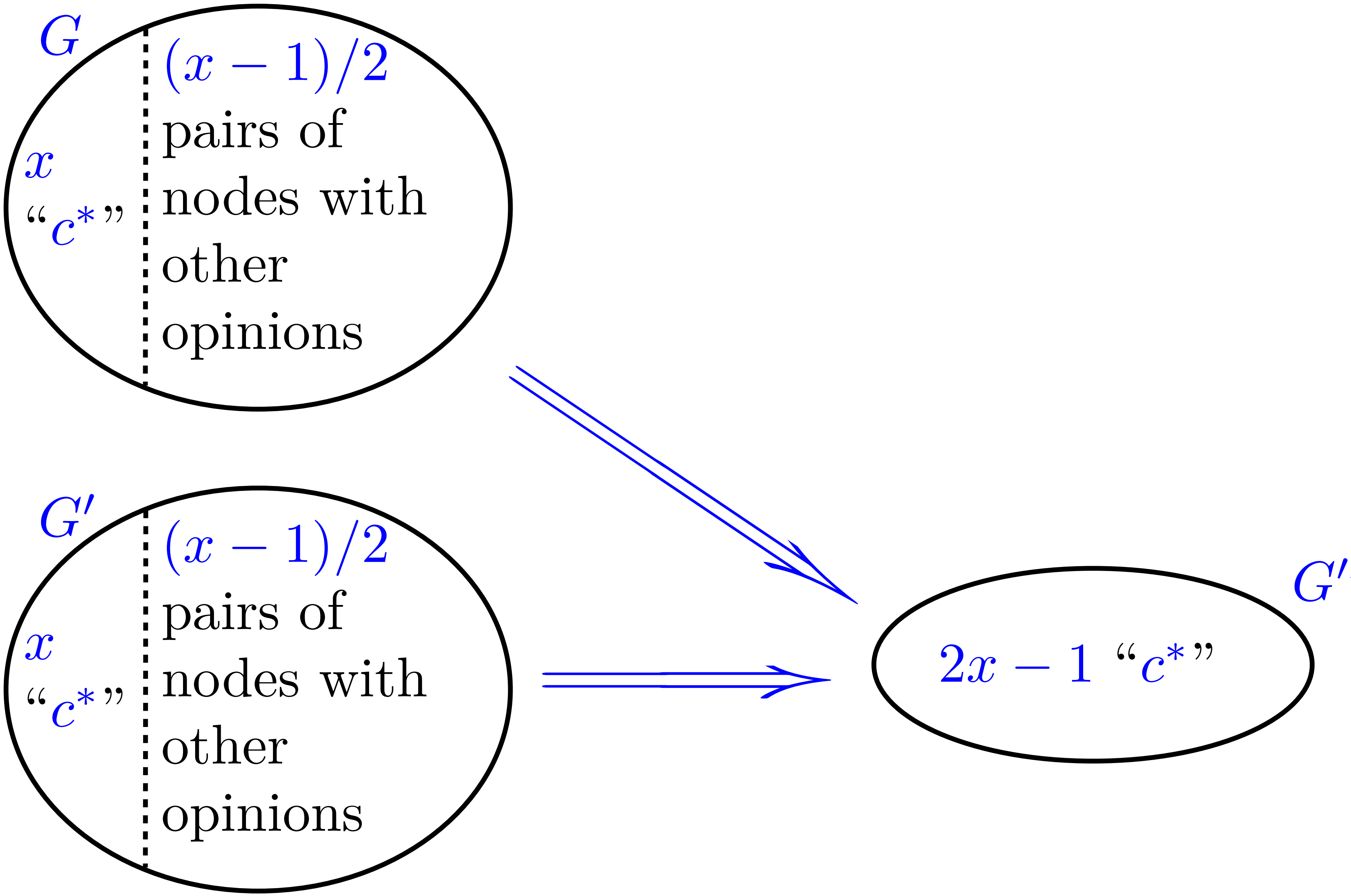
Pigeonhole: if $|\Sigma| < k^2 - k$, 2 config.s G and G' in
converge to identical configurations.

$\Omega(k^2)$ Lower Bound II



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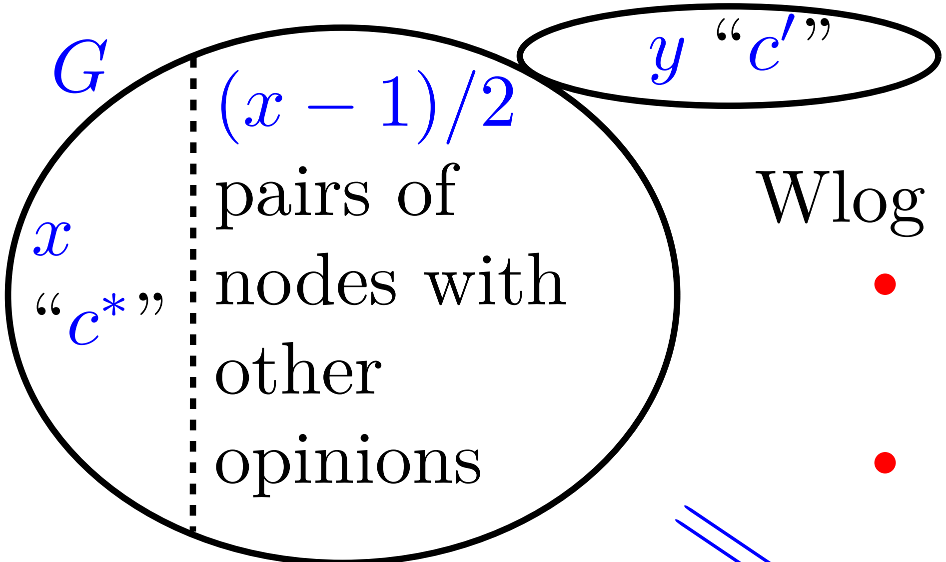
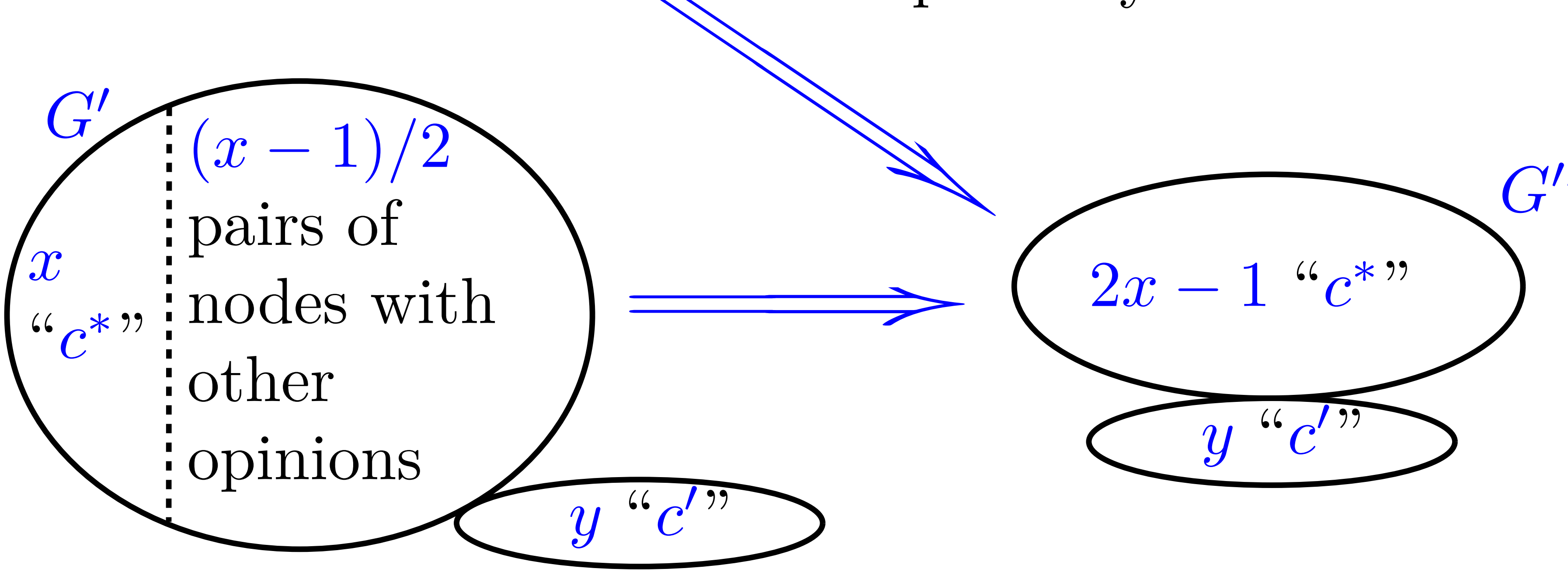


Wlog there is opinion c' and y s.t.

- adding y "c'" to G makes c' plurality in G
- adding y "c' to G' leaves c^* plurality in G'.



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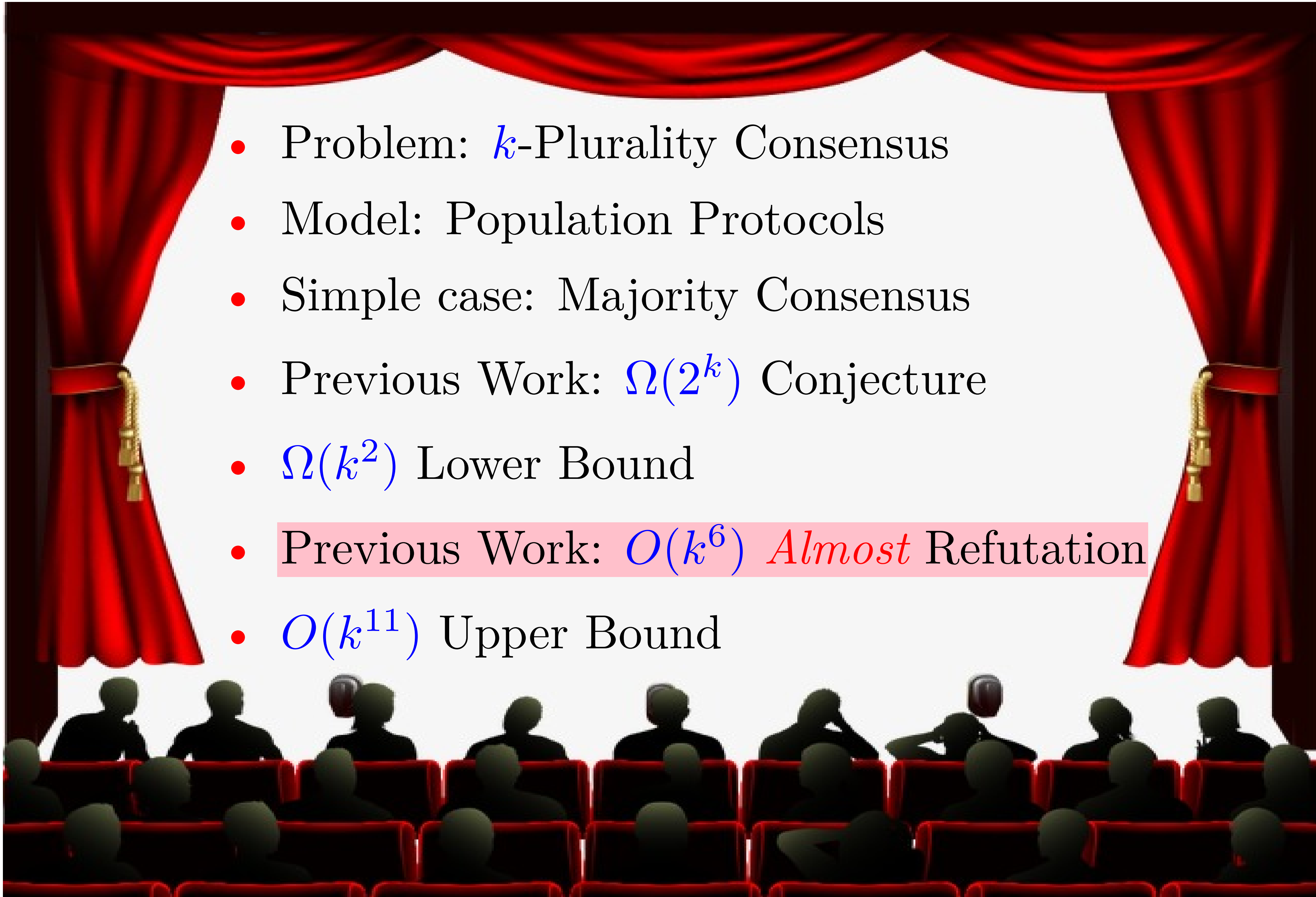

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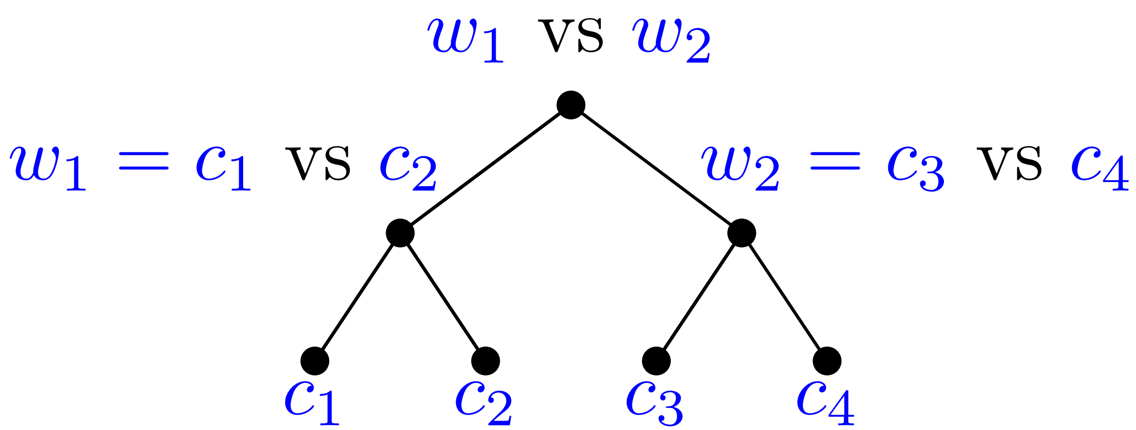
Plurality Consensus via Tournament Tree

Idea. Compute plurality by *majority* tournament.



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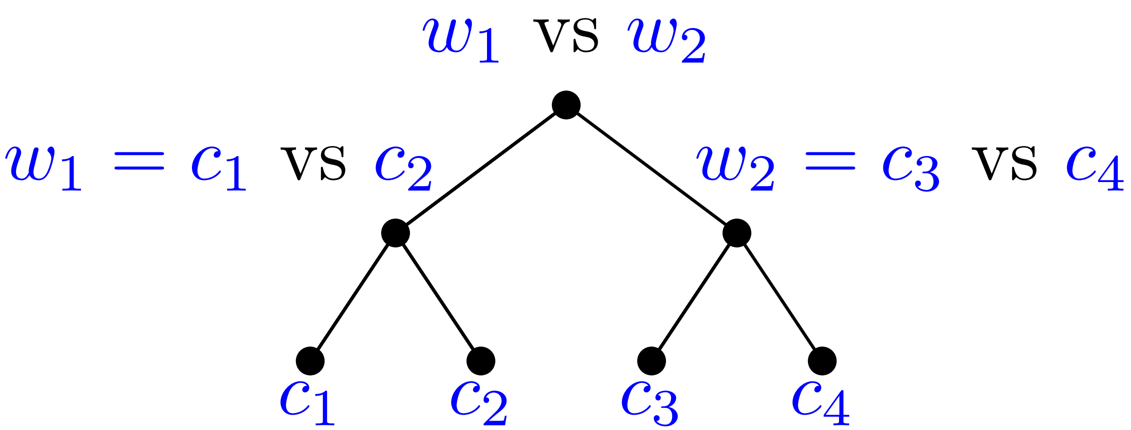

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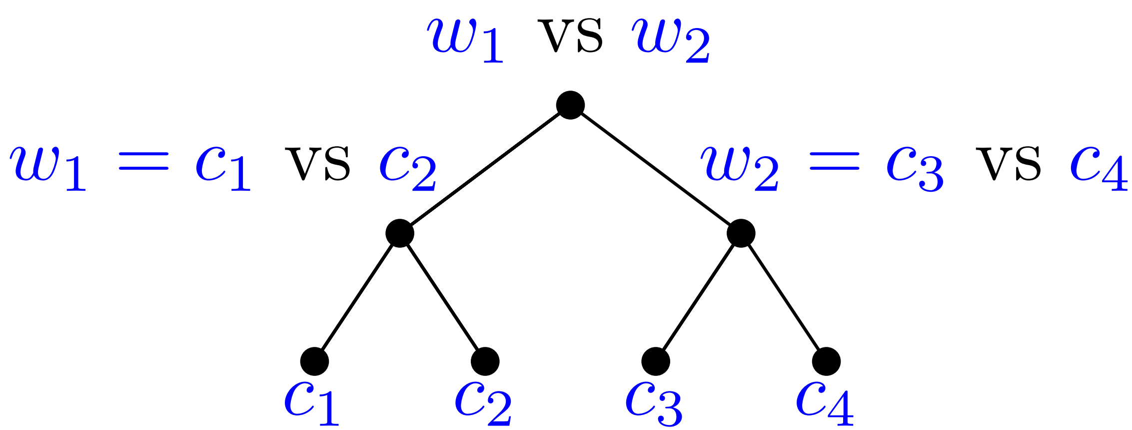

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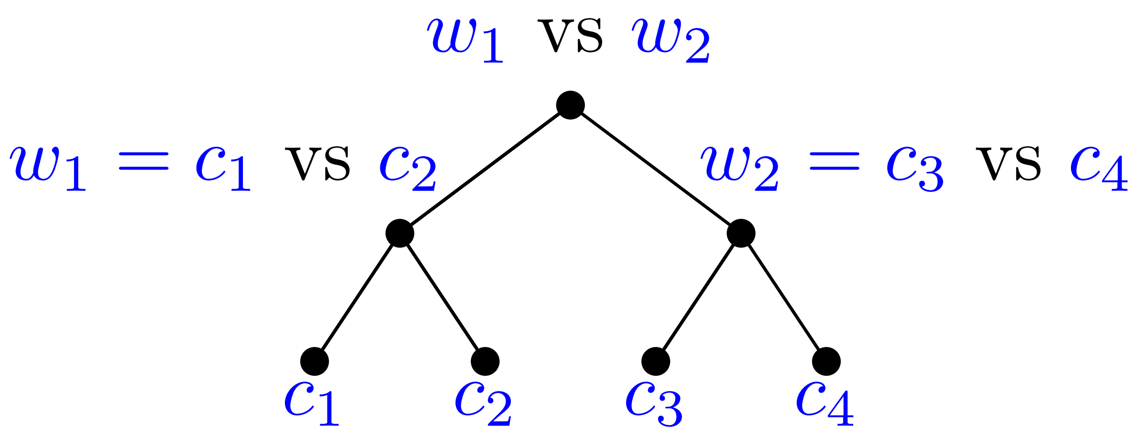

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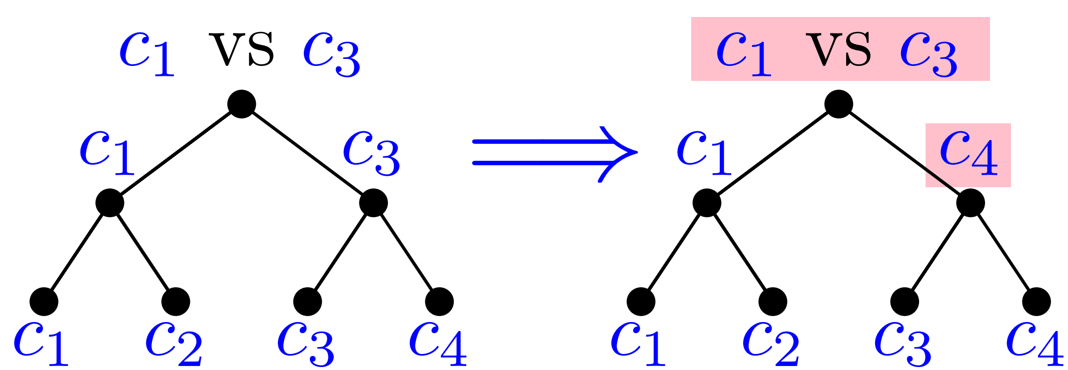

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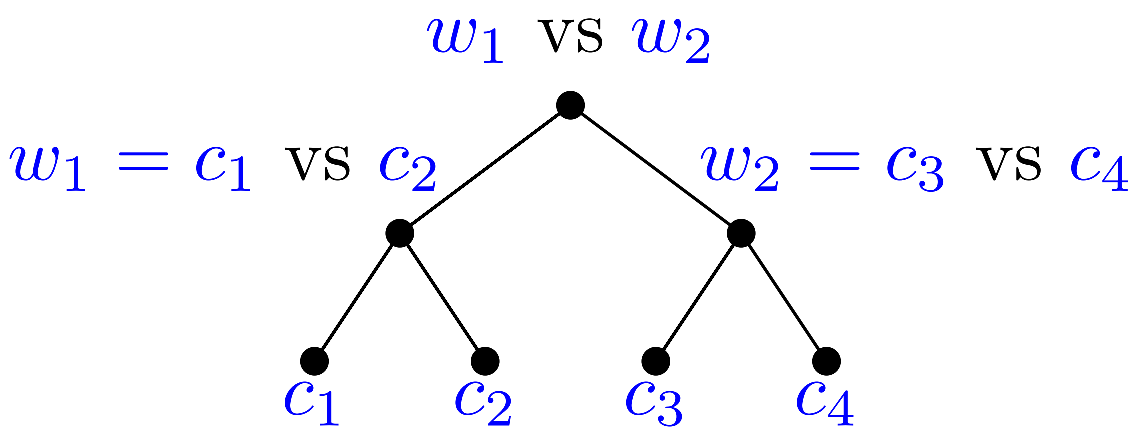

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Solved if nodes can change *opinion*.

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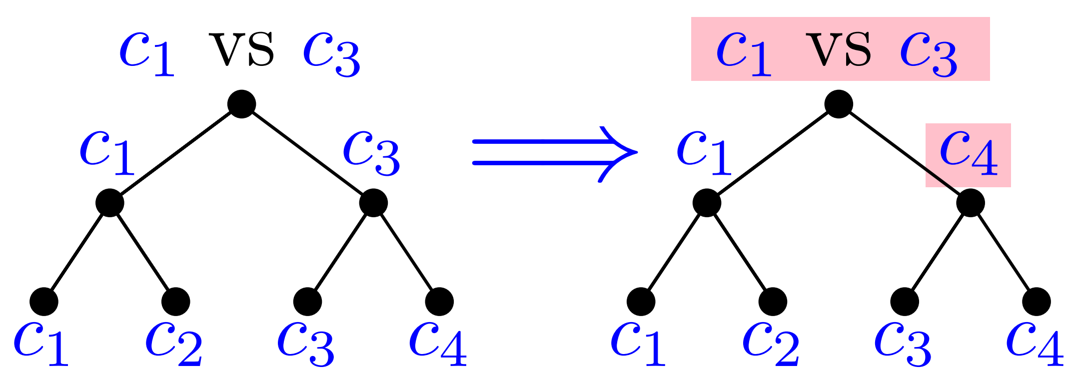

Dynamic Plurality Consensus

[Gasieniec et al. OPODIS'16]

Nodes can *change opinion* during execution.

States and weights

s	$w(s)$
$[-2]$	-2
$[-1]$	-1
$[0], \langle -1 \rangle, \langle 0 \rangle, \langle 1 \rangle$	0
$[1]$	1
$[2]$	2

Updating the state

s_a, $c_a = 1$ changes to $c'_a = -1$	s'_a
$[0], \langle -1 \rangle, \langle 0 \rangle, \langle 1 \rangle$	$[-2]$
$[1]$	$[-1]$
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$[-2]$	$[0]$

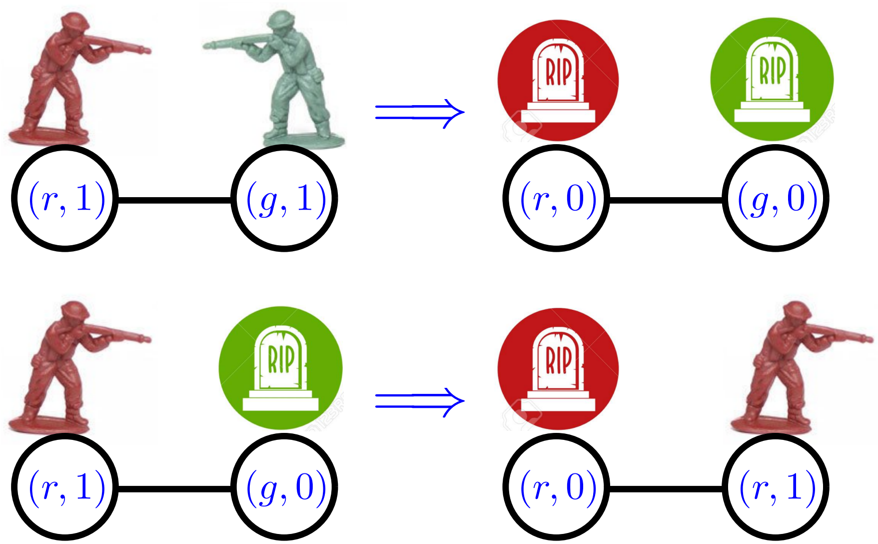
Transitions

$s_a \backslash s_b$	$[-2]$	$[-1]$	$[0]$	$[1]$	$[2]$
$[-2]$	$([-2], [-2])$	$([-2], [-1])$	$([-2], \langle -1 \rangle)$	$([-1], \langle -1 \rangle)$	$([0], [0])$
$[-1]$	$([-1], [-2])$	$([-1], [-1])$	$([-1], \langle -1 \rangle)$	$([0], [0])$	$(\langle 1 \rangle, [1])$
$[0]$	$(\langle -1 \rangle, [-2])$	$(\langle -1 \rangle, [-1])$	$([0], [0])$	$(\langle 1 \rangle, [1])$	$(\langle 1 \rangle, [2])$
$[1]$	$(\langle -1 \rangle, [-1])$	$([0], [0])$	$(\langle 1 \rangle, [1])$	$([1], [1])$	$([1], [2])$
$[2]$	$([0], [0])$	$([1], \langle 1 \rangle)$	$([2], \langle 1 \rangle)$	$([2], [1])$	$([2], [2])$
weak	$(\langle -1 \rangle, [-2])$	$(\langle -1 \rangle, [-1])$	$(\langle 0 \rangle, [0])$	$(\langle 1 \rangle, [1])$	$(\langle 1 \rangle, [2])$

Dynamic Plurality Consensus

[Gasieniec et al. OPODIS'16]

Nodes can *change opinion* during execution.



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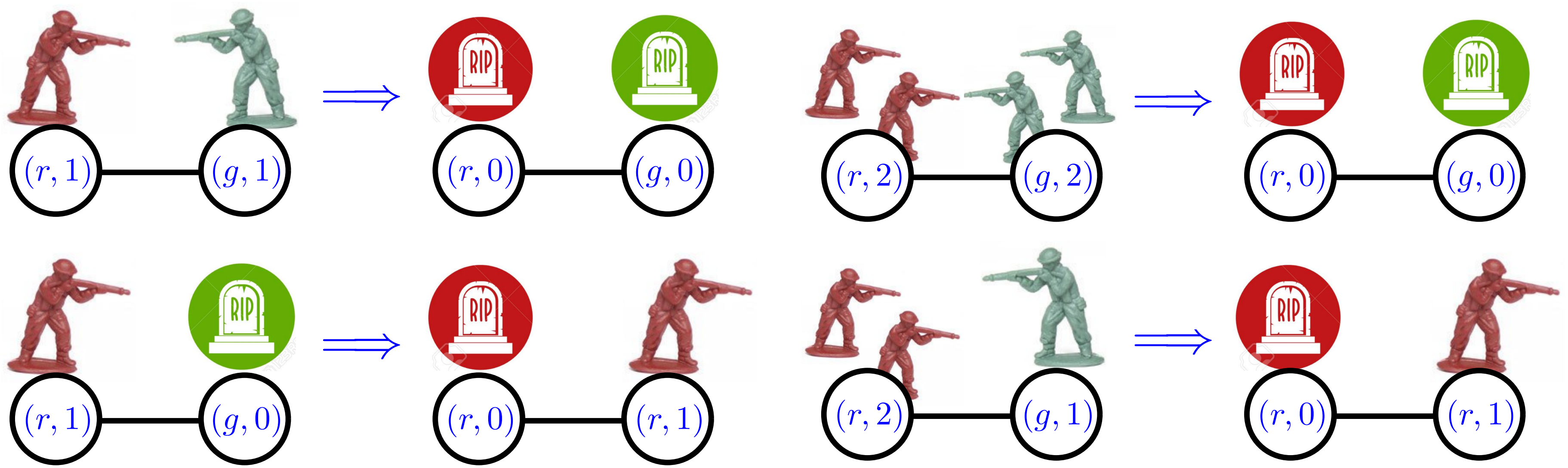

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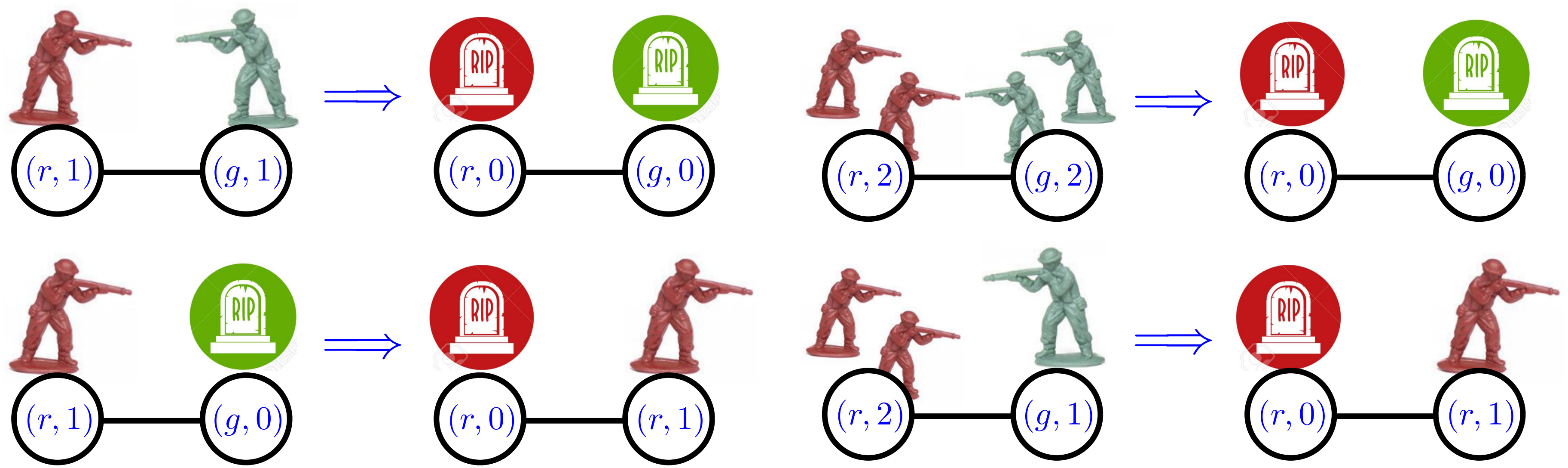

Nodes changing opinion generate *two* soldiers of the new opinion.

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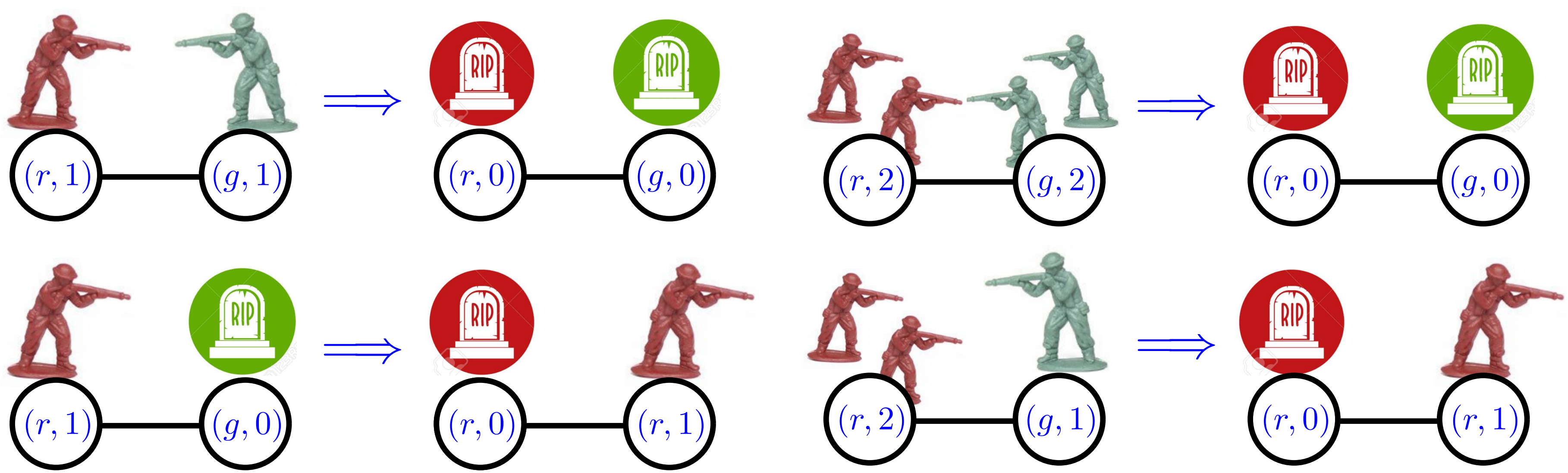

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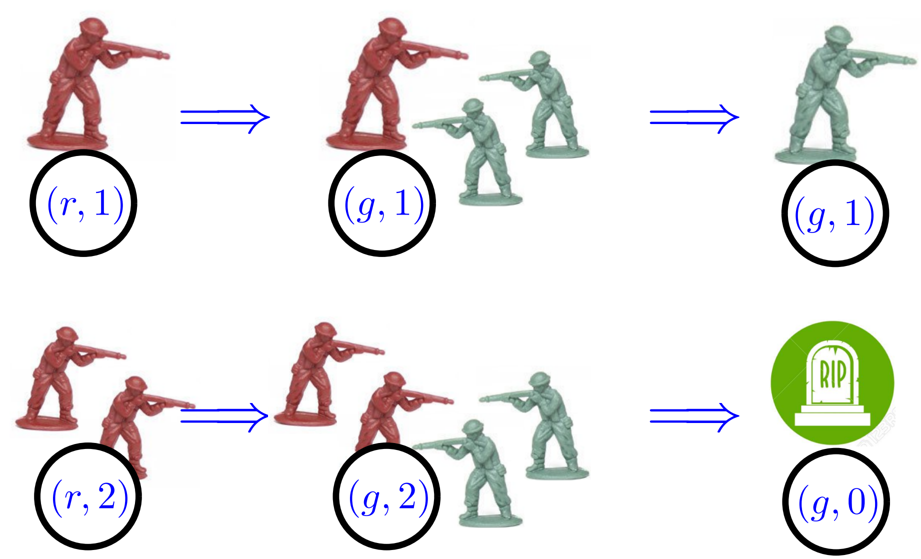

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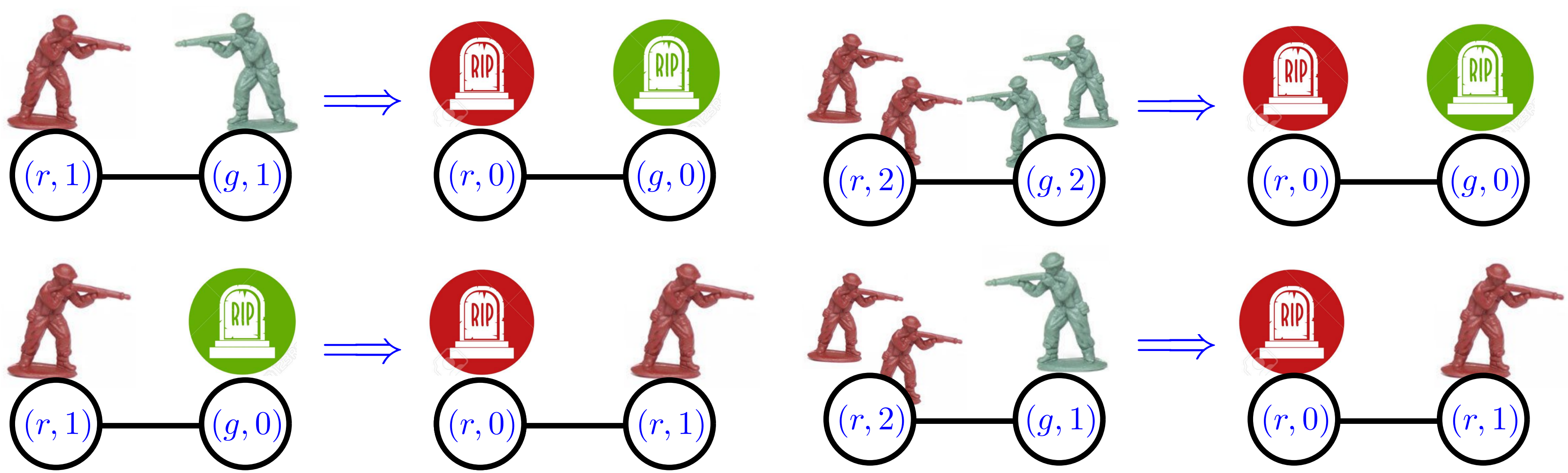

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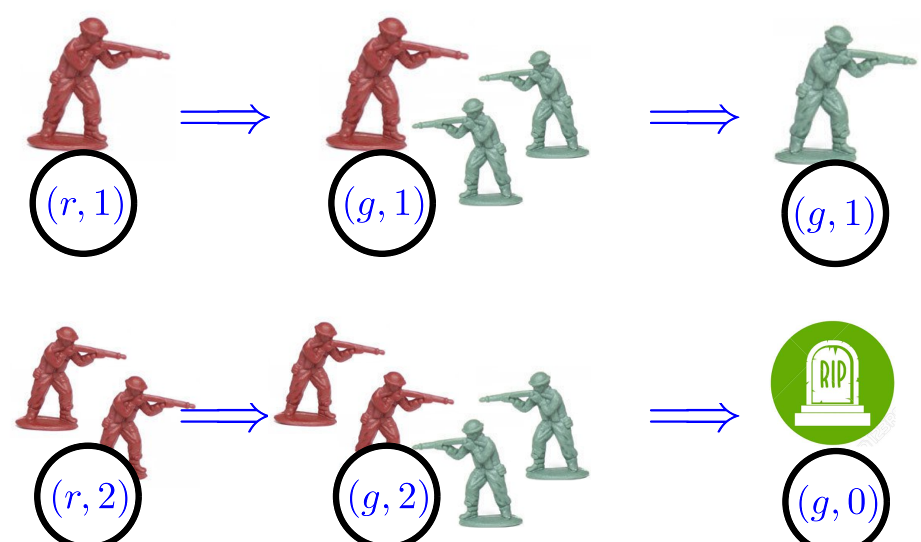
Balance of opinions
equals
balance of soldiers



Outline

- Problem: k-Plurality Consensus
- Model: Population Protocols
- Simple case: Majority Consensus
- Previous Work: $\Omega(2^k)$ Conjecture
- $\Omega(k^2)$ Lower Bound
- Previous Work: $O(k^6)$ *Almost* Refutation
- $O(k^{11})$ Upper Bound

$O(k^{11})$ Upper Bound (Refunting Conjecture)

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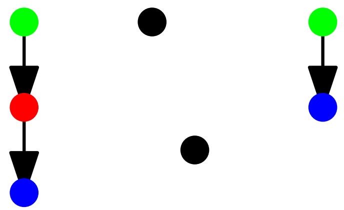
Problem. Multiple lists can appears. How to delete/merge lists?



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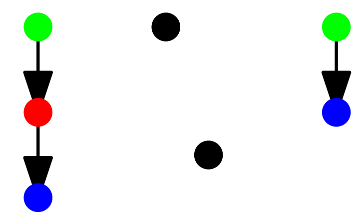
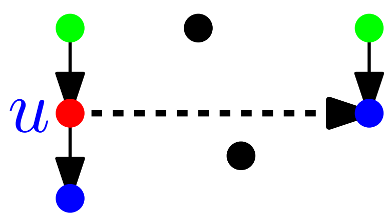
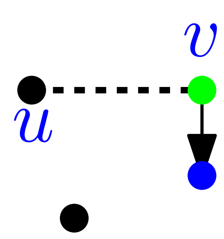


Ideas. Start deleting from *roots* of lists and append elements by travelling from root to last item.



u will inform parent that list shall be deleted.



u starts by designating v as parent. Eventually u designates as parents v's child, and so on.

Conclusions & Open Problem

Non-ordered self-stabilizing plurality consensus in population protocols with fair scheduler can be solved using $O(k^{11})$ states per agent.

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What is the space complexity of plurality consensus in population protocols with fair scheduler?

Project Idea

Simulate the DMVR and Dynamics Majority algorithms on Erdős-Rényi graphs with parameter p, varying the parameter.

- Salehkaleybar, S., A. Sharif-Nassab, and S.J. Golestani. 2015. "Distributed Voting/Ranking with Optimal Number of States per Node." IEEE Transactions on Signal and Information Processing over Networks PP (99): 1–1. https://doi.org/10.1109/TSIPN.2015.2477777.
- Gasieniec, Leszek, David Hamilton, Russell Martin, Paul G. Spirakis, and Grzegorz Stachowiak. 2017. "Deterministic Population Protocols for Exact Majority and Plurality." In 20th International Conference on Principles of Distributed Systems (OPODIS 2016), 70:14:1–14:14. Leibniz International Proceedings in Informatics (LIPIcs). https://doi.org/10.4230/LIPIcs.OPODIS.2016.14.

Simulations should be performed using open-source software with some effort to make them efficient (e.g. coded in Python using Numpy), and the source code should be made publicly available (e.g. on Gitlab) and GPL licensed.