Stabilizing Consensus with Many Opinions

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joint work with
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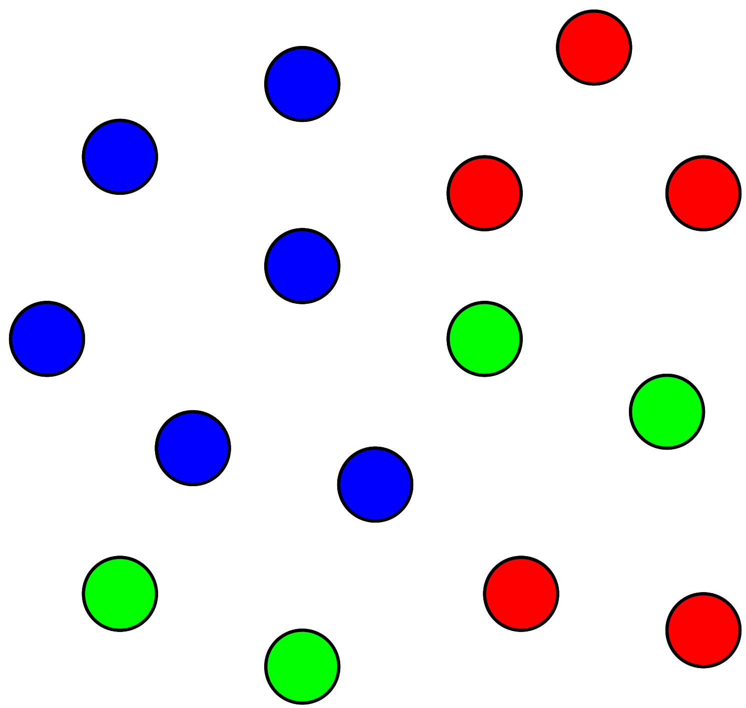
Stabilizing Almost-Consensus

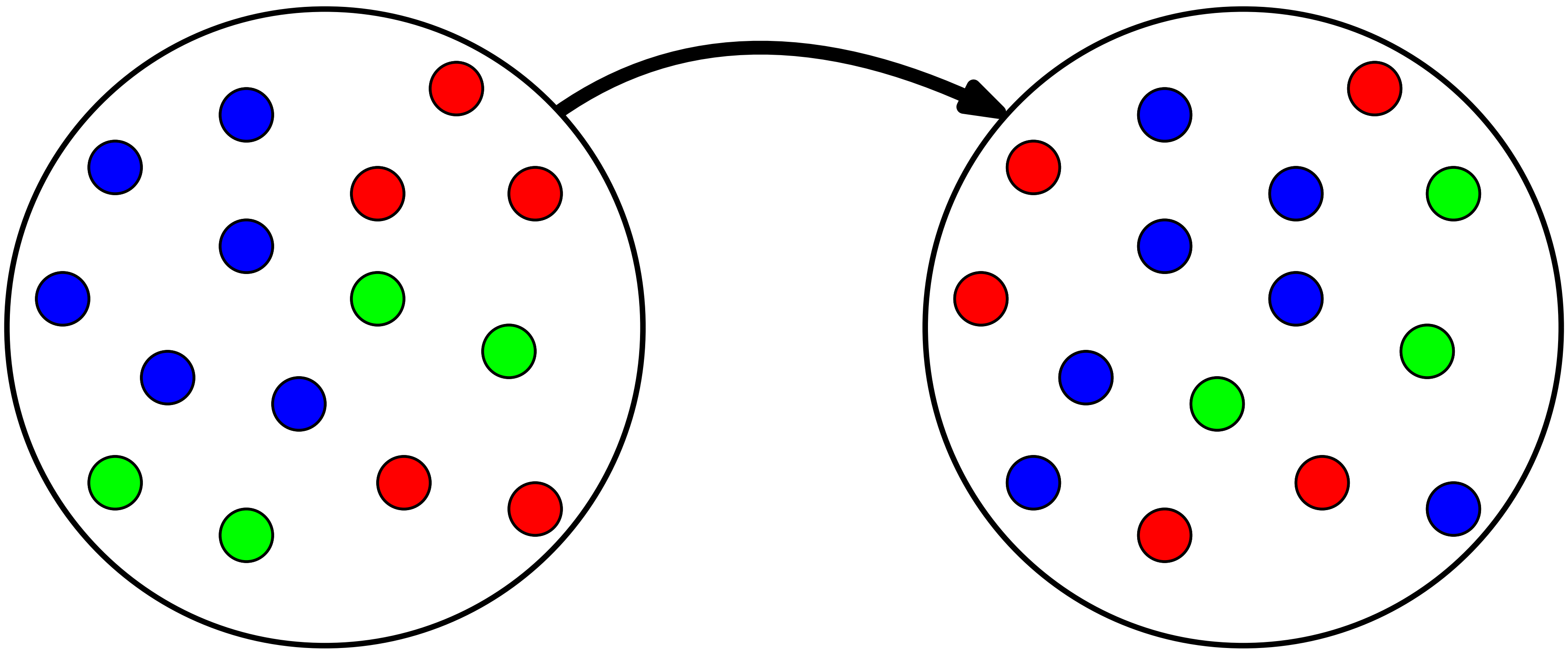
A set of nodes each having one color out of a set Σ.



Initial colors are called *valid*.

Stabilizing Almost-Consensus

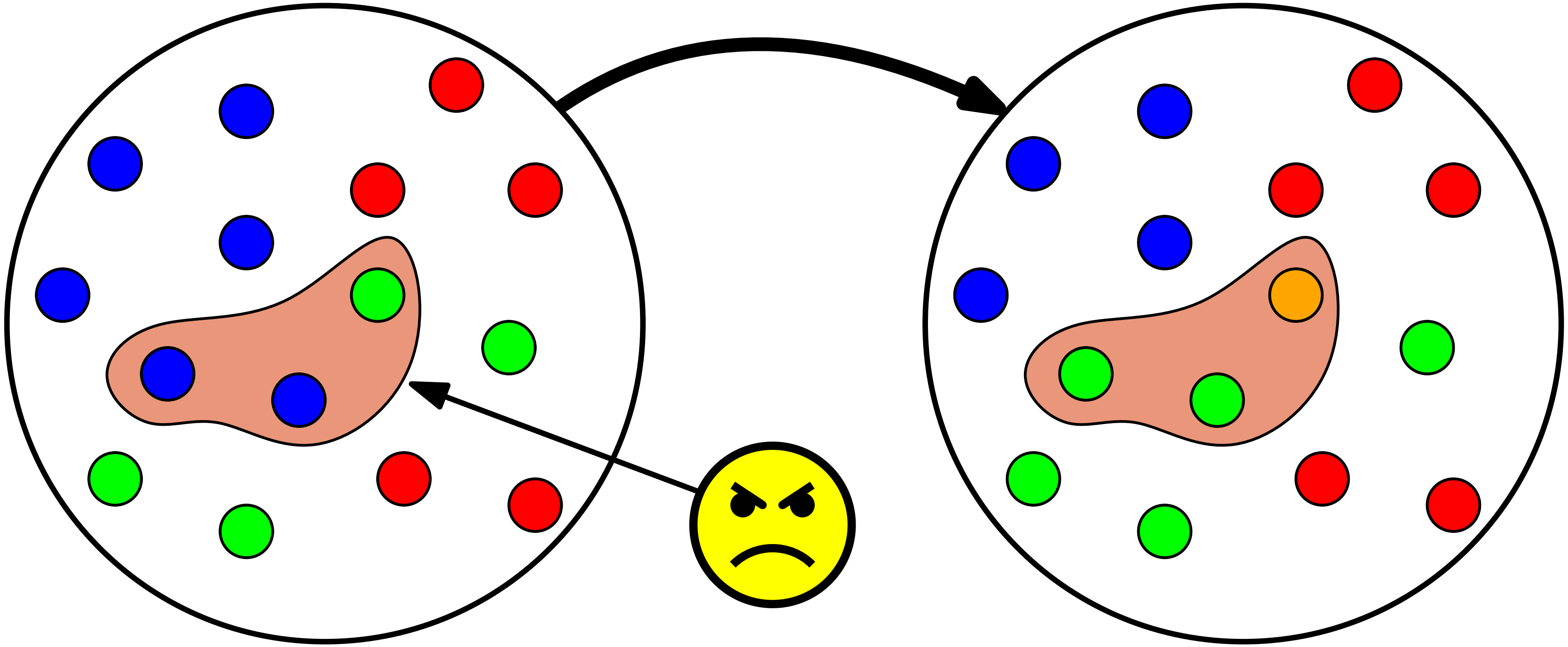
At the start of each round, nodes uptate their color,



according to the given communication model and protocol.

Stabilizing Almost-Consensus

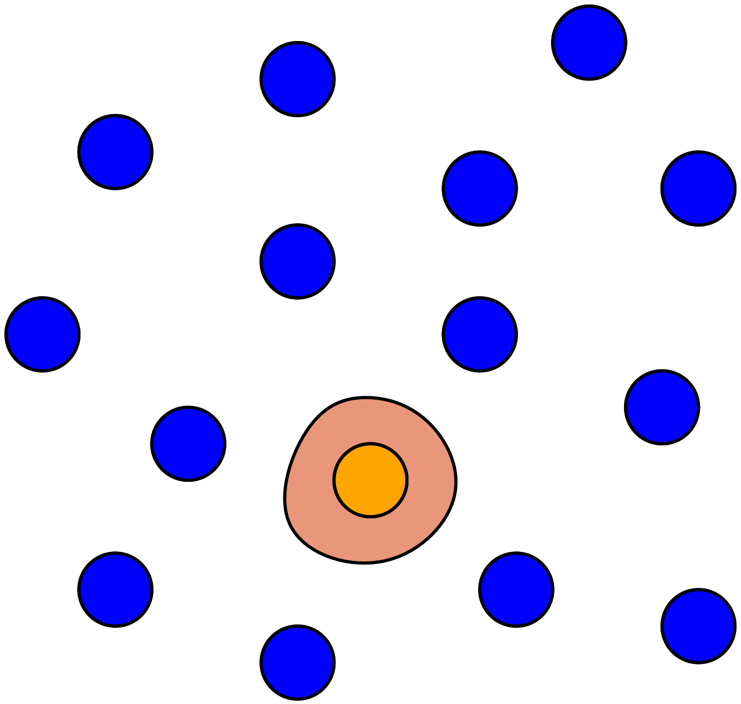
At the end of each round, an F-dynamic adversary can change the color of F nodes,



possibly chosing different subsets of nodes over different rounds.

Stabilizing Almost-Consensus

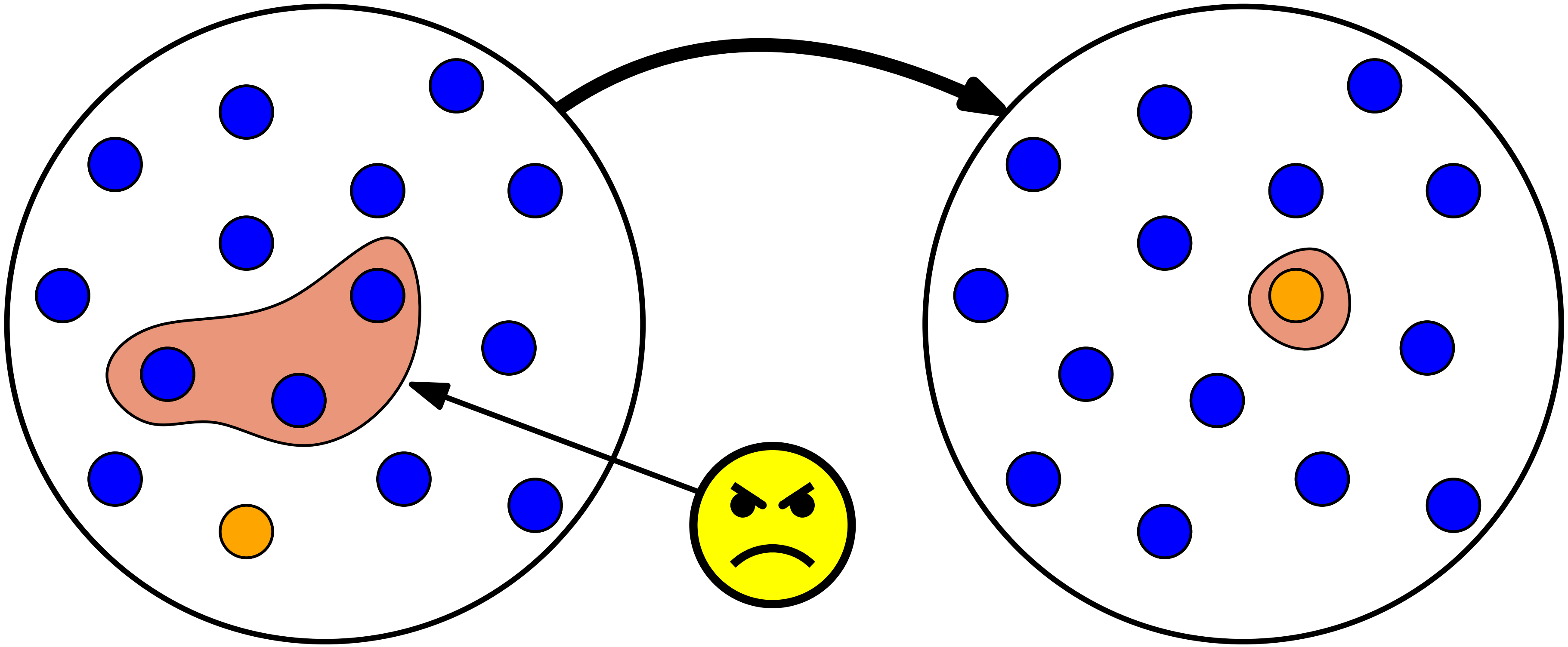
Except for a small number of nodes
we want to reach consensus (almost consensus),



on any valid color (almost validity),

Stabilizing Almost-Consensus

Almost-consensus has to be preserved for any poly(n) rounds,



even if the adversary changes colors at each round (almost stability).

Stabilizing Almost-Consensus

A *stabilizing almost-consensus* protocol guarantees that, w.h.p., for some $\gamma < 1$, from any initial conf., in a finite number of rounds, the system reaches a set of conf.s where $n - \mathcal{O}(n^{\gamma})$ nodes

- hold the same color (*almost consensus*),
- the color was in the initial conf. (*almost validity*),
- and the convergence hold for any poly(n) rounds (*almost stability*).

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Cf. classical byzantine agreement:
agreement, validity and termination.

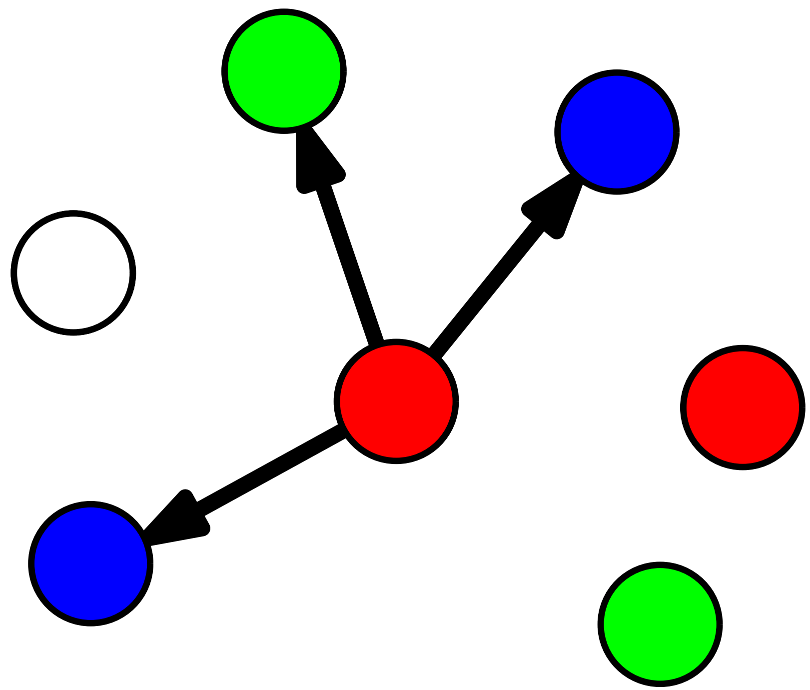
The Setting

Communication model. Uniform Gossip model: Each node in one round can communicate with one node chosen u.a.r.

Protocol constraints. *simple* rule (dynamics): Anonymous, $O(\log |\Sigma|)$ local memory and message size, counters of non-constant length, ...

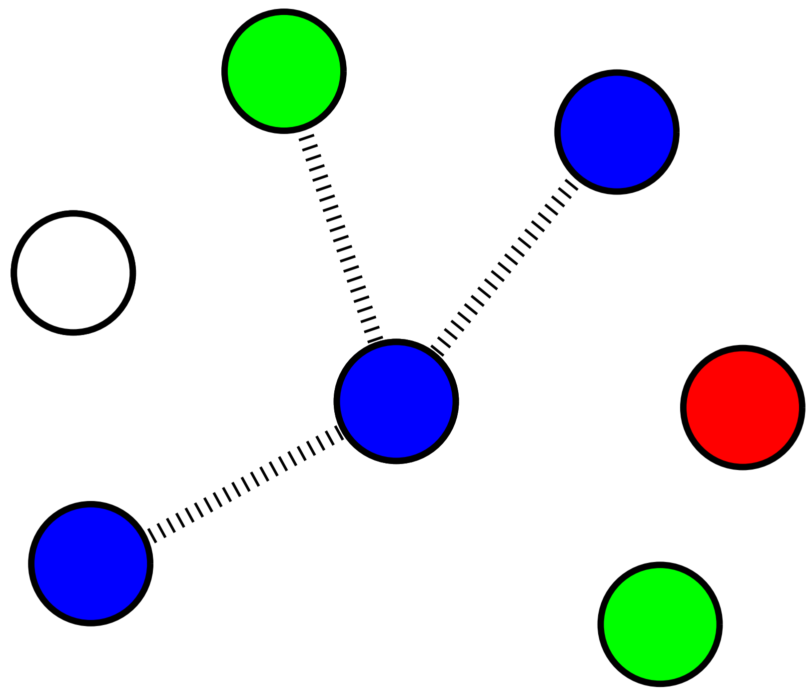
Motivations. Biological systems, chemical reaction networks, social networks, sensor networks.

Previous Work: 3-Median Dynamics



Each node observes the color of three other nodes chosen u.a.r....

Previous Work: 3-Median Dynamics



...and changes its color according to the median of these three...

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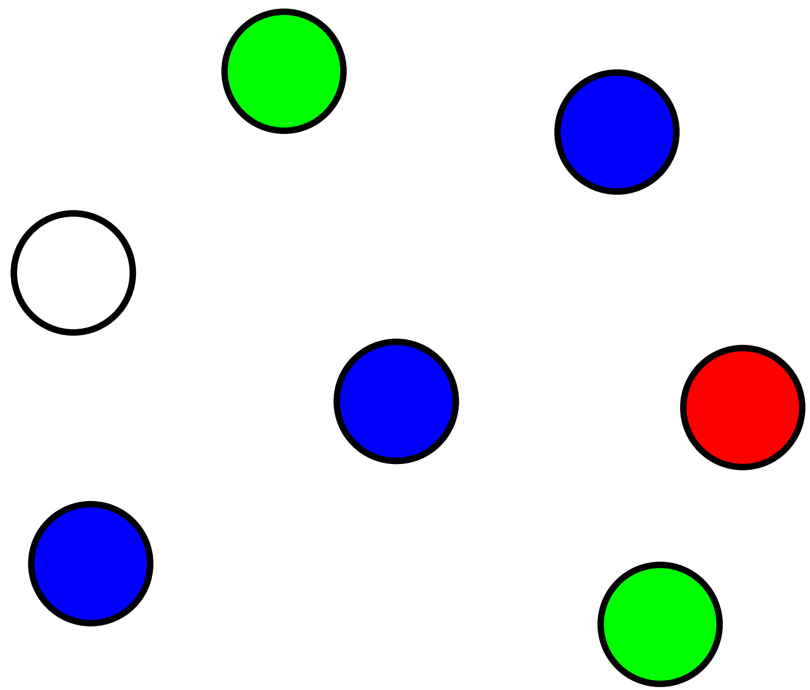
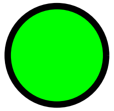
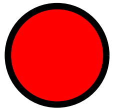
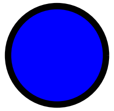
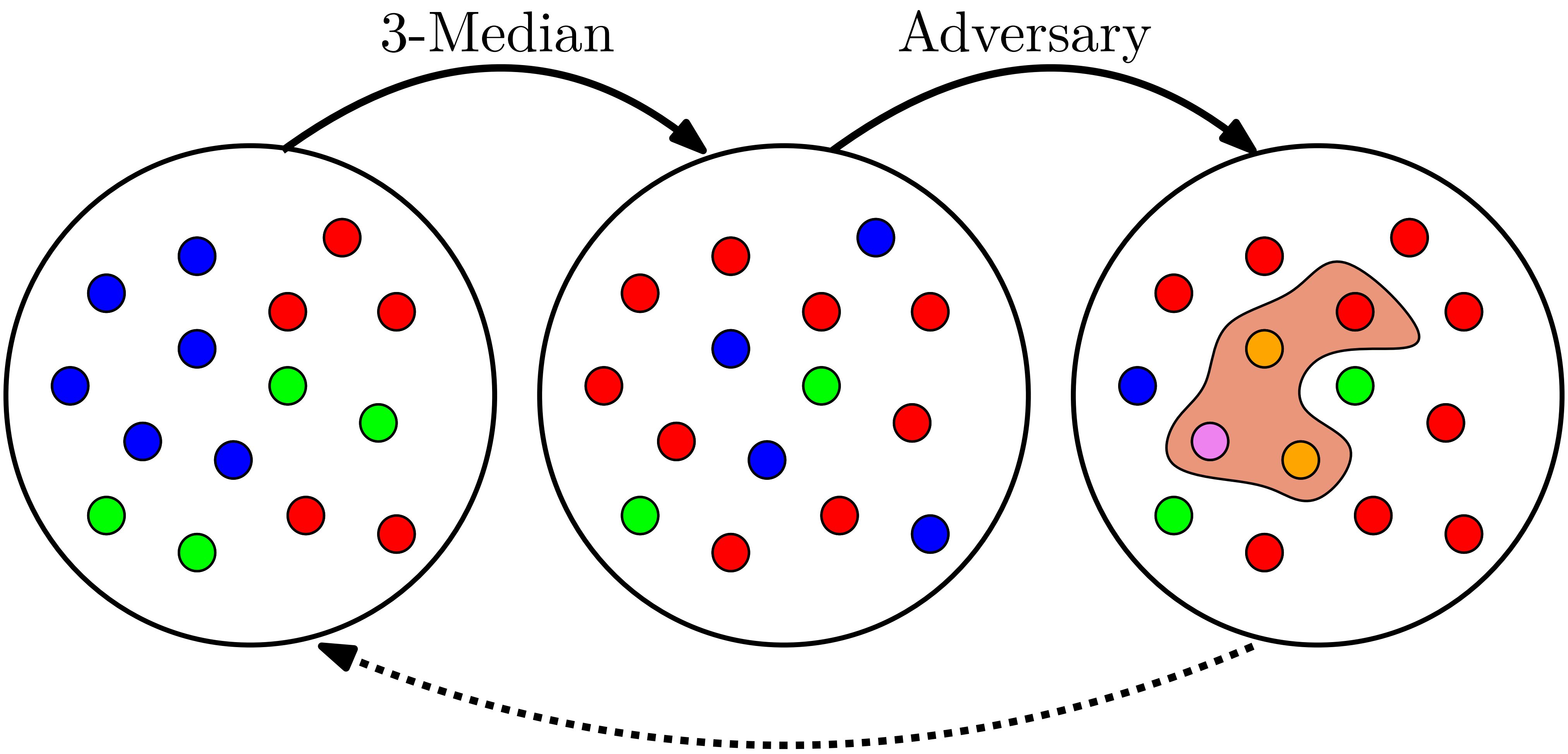


Colors are totally ordered: ...<  <  <  <...

The 3-Median Process



Almost consensus? Almost validity? Almost stability?

3-Median Dynamics

Theorem (Doerr, Goldberg, Minder, Sauerwald, Scheideler '11). For any $\sqrt{n}$-bounded adversary, the **3-median** computes an almost stable value between the $(n/2 - c\sqrt{nlogn})$-largest and the $(n/2 + c\sqrt{nlogn})$- largest of the initial values, in $\mathcal{O}(\log k \cdot \log \log n + \log n)$ rounds w.h.p.

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Does 3-median guarantee Stabilizing Almost Consensus?

- Almost consensus ✓
- Almost validity
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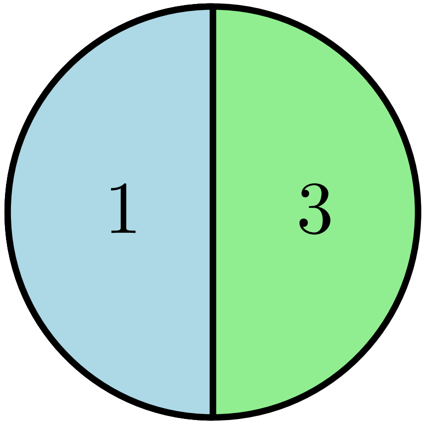

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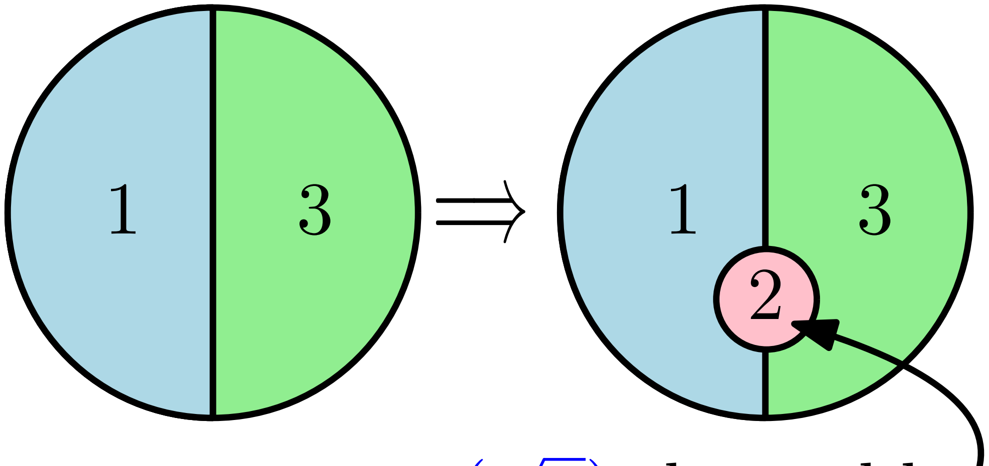
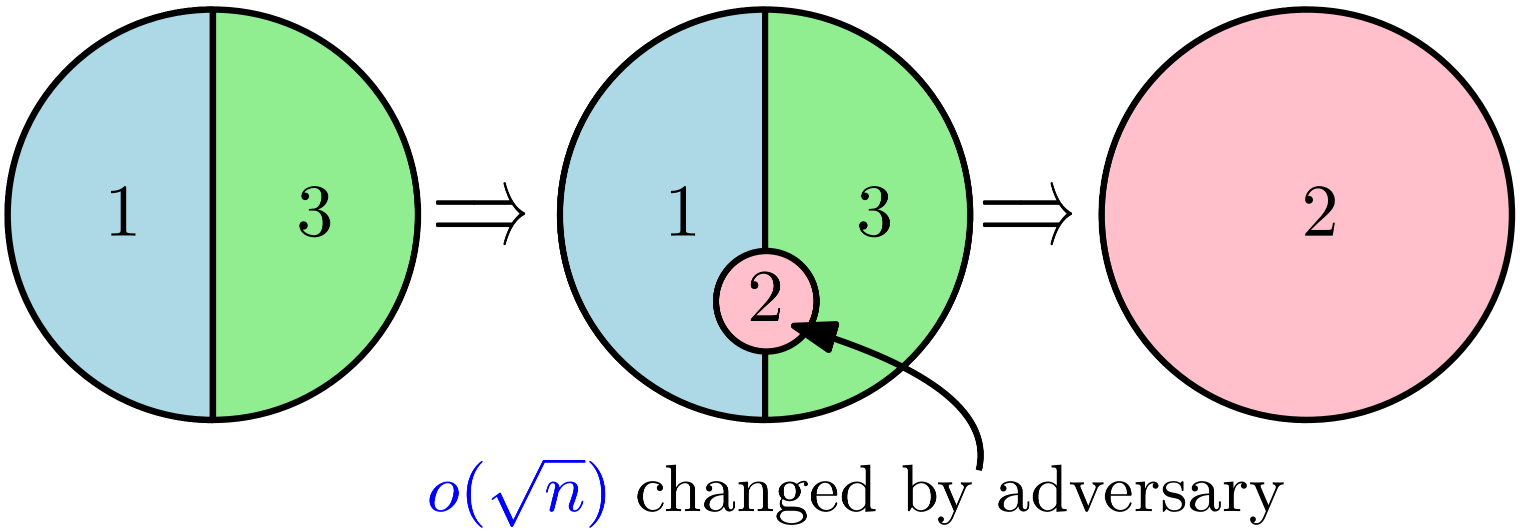


$o(\sqrt{n})$ changed by adversary

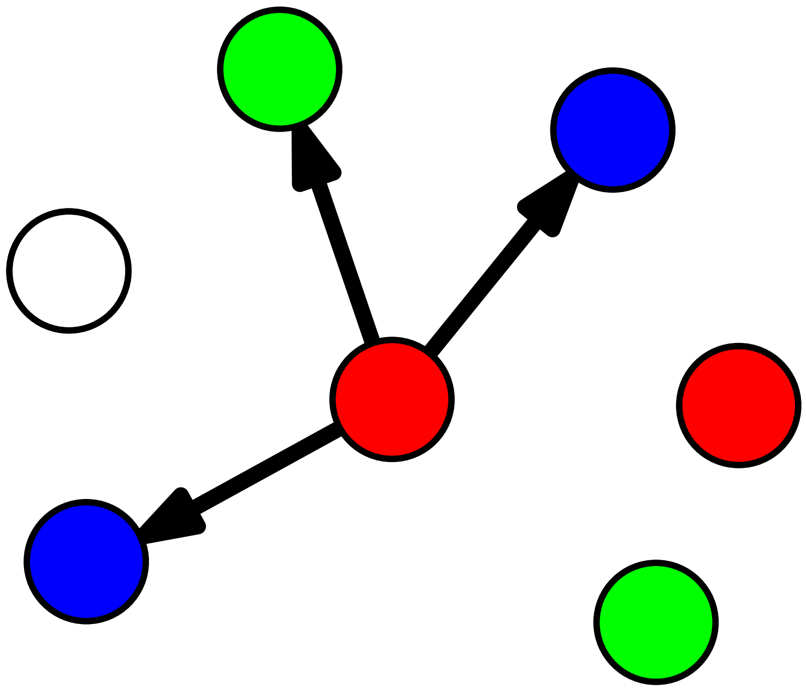
3-Median Dynamics

No almost validity!



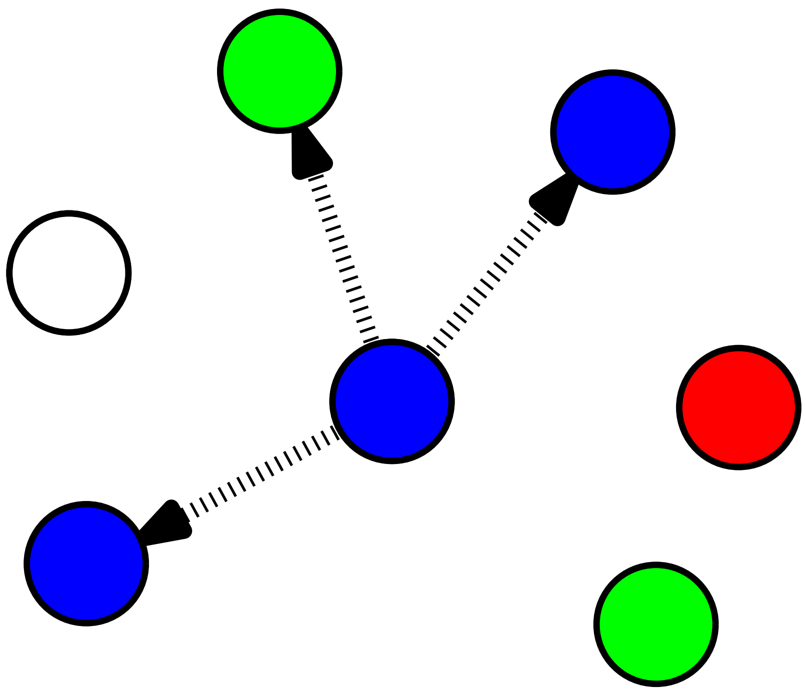
The adversary can manipulate the system.

The 3-Majority Dynamics



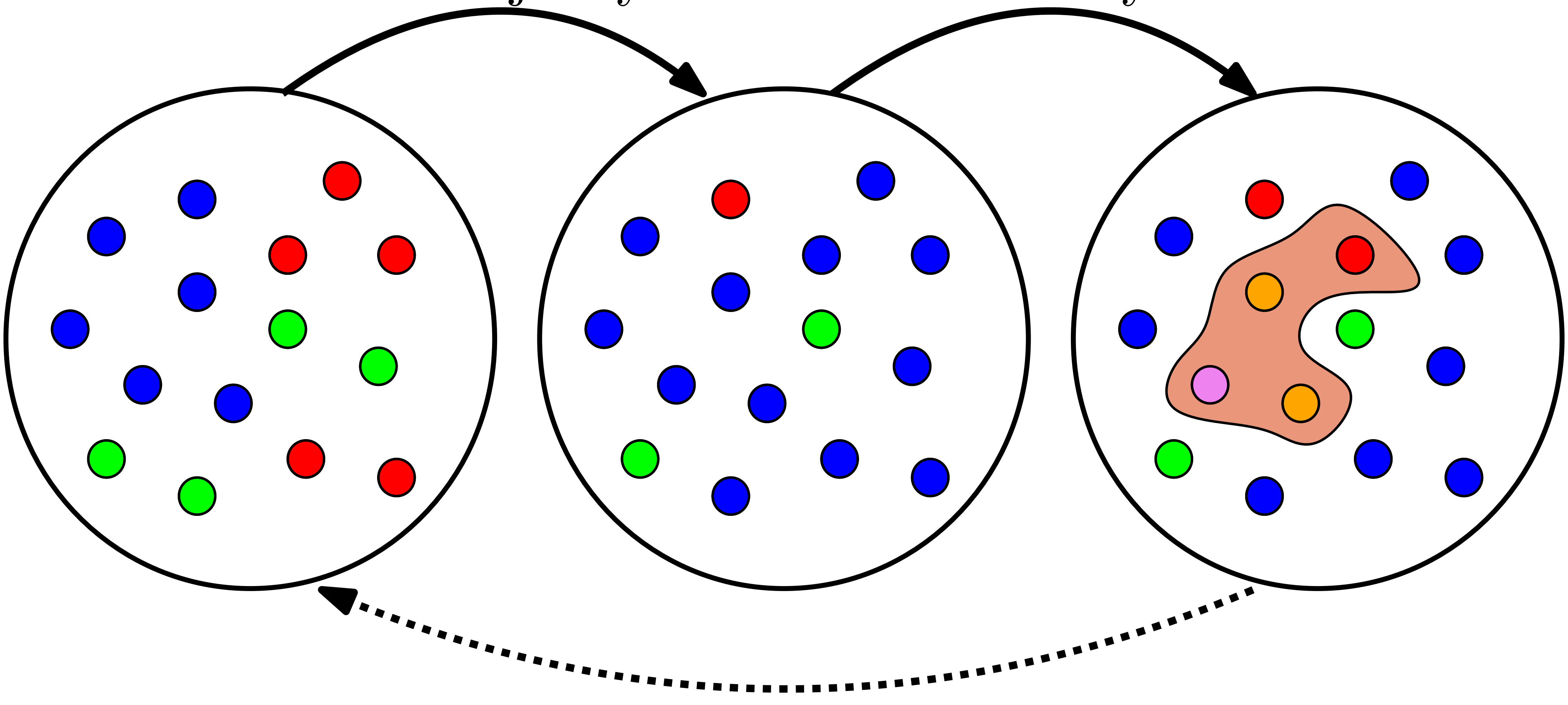
Each node observes the color of three other nodes chosen u.a.r....

The 3-Majority Dynamics



...and changes its color according to the majority of these three (breaking ties u.a.r.).

The Majority Process



Almost consensus? Almost validity? Almost stability?

3-Majority for Plurality Consensus

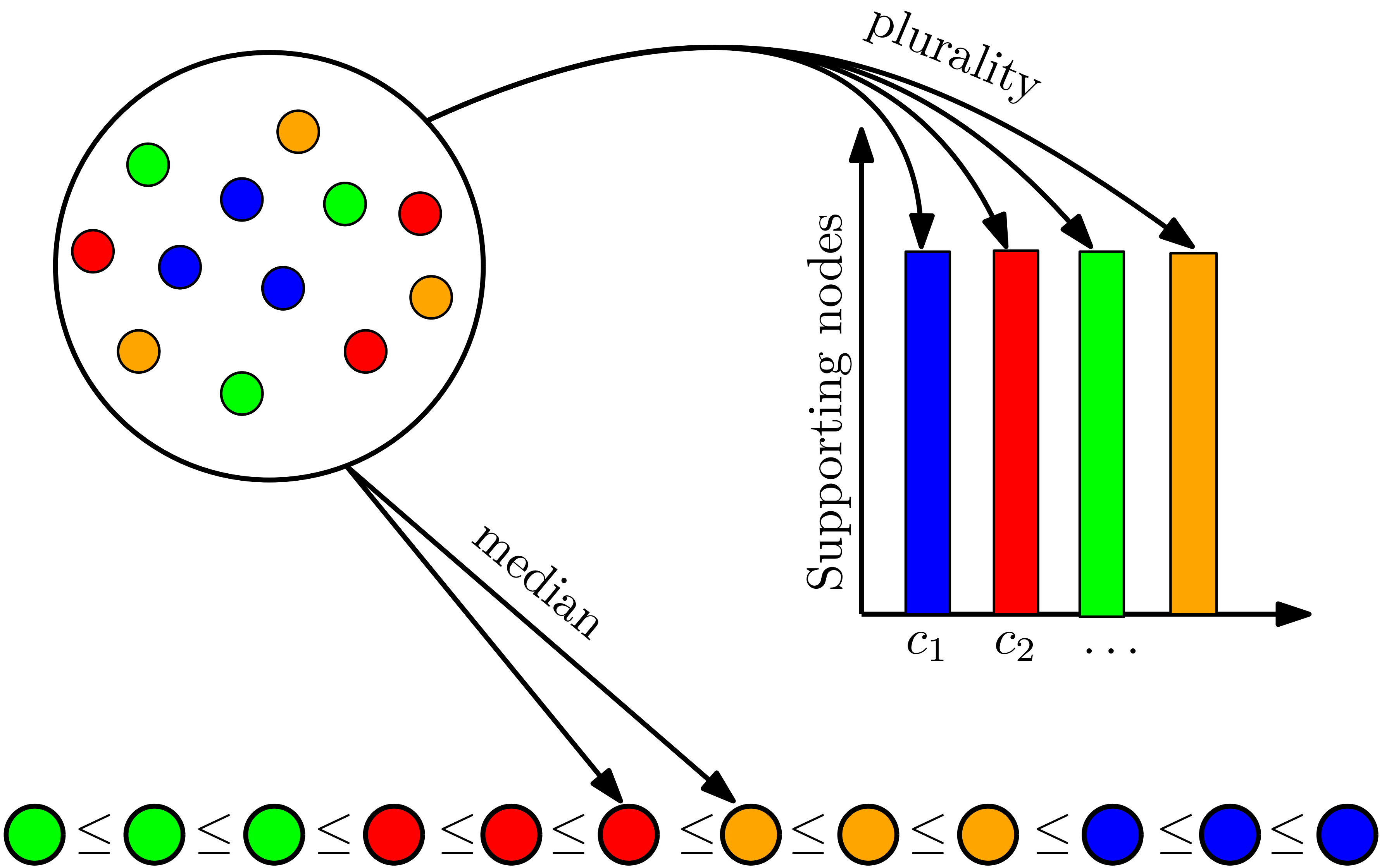
$c_i^{(t)} := |\{i\text{-colored nodes}\}|$ color 1 is the plurality

Initial bias s: For all $i \neq 1$, $c_1 - c_i \geq s$

Theorem (Becchetti, Clementi, Natale, Pasquale, Silvestri, Trevisan '14).

- From any configuration with $k < \sqrt[3]{n}$ colors, with bias $s = \Omega(\sqrt{kn \log n})$, the **3-majority** converges to the plurality color in $O(k \log n)$ rounds w.h.p., against a $O(\sqrt{n})$-bounded dynamic adversary.
- From configurations where every color is supported by almost n/k nodes, convergence takes $\Omega(k)$ rounds w.h.p.

3-Majority vs 3-Median



3-Majority with Bias

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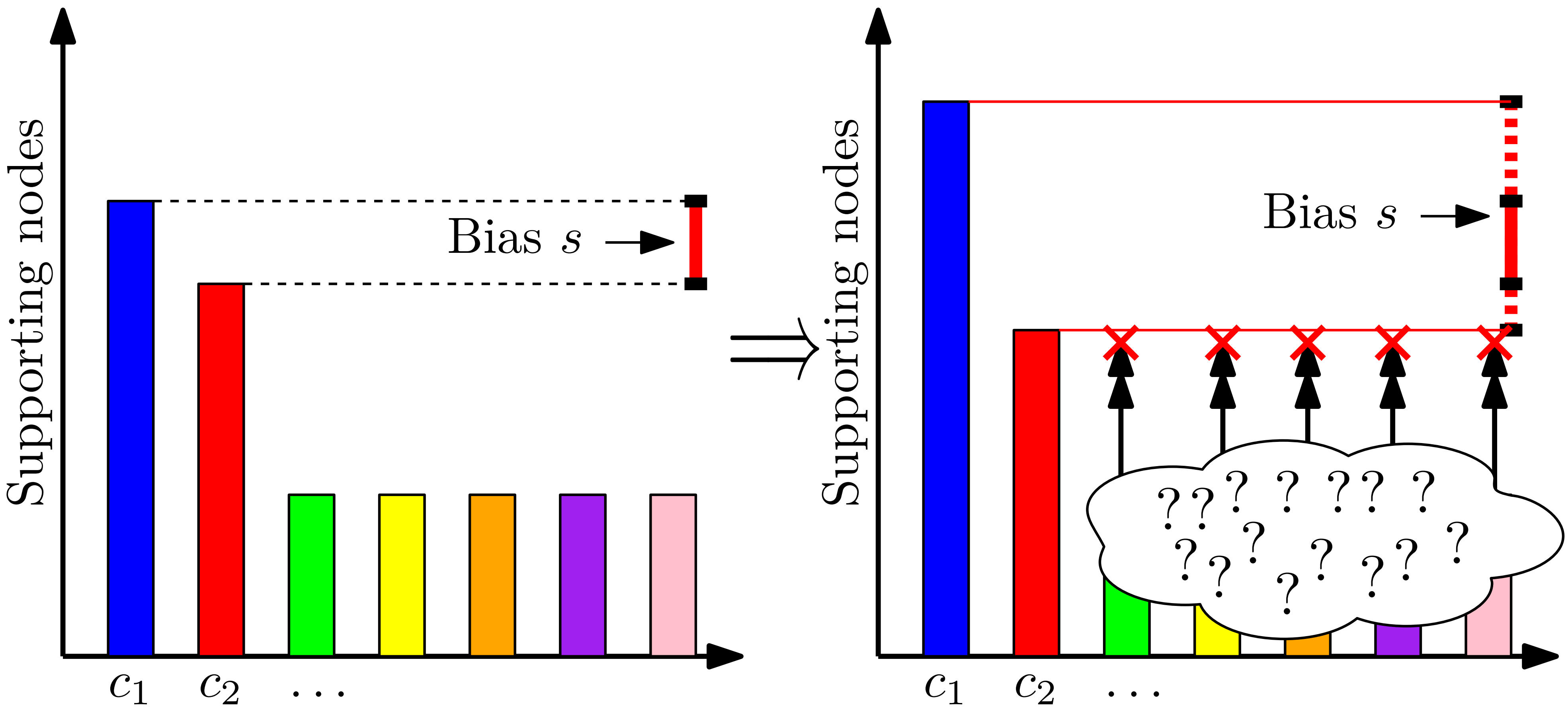

Our Contribution: 3-Majority without Bias

What if we start from any initial configuration, i.e. there may be no initial bias?

Theorem. Let $k \leq n^{\alpha}$, for a suitable constant $\alpha < 1$, and $F = \mathcal{O}(\sqrt{n}/(k^{\frac{5}{2}} \log n))$. The 3-majority dynamics is a stabilizing almost-consensus protocol against any F-dynamic adversary, with convergence time $\mathcal{O}((k^2\sqrt{\log n} + k \log n)(k + \log n))$, w.h.p.

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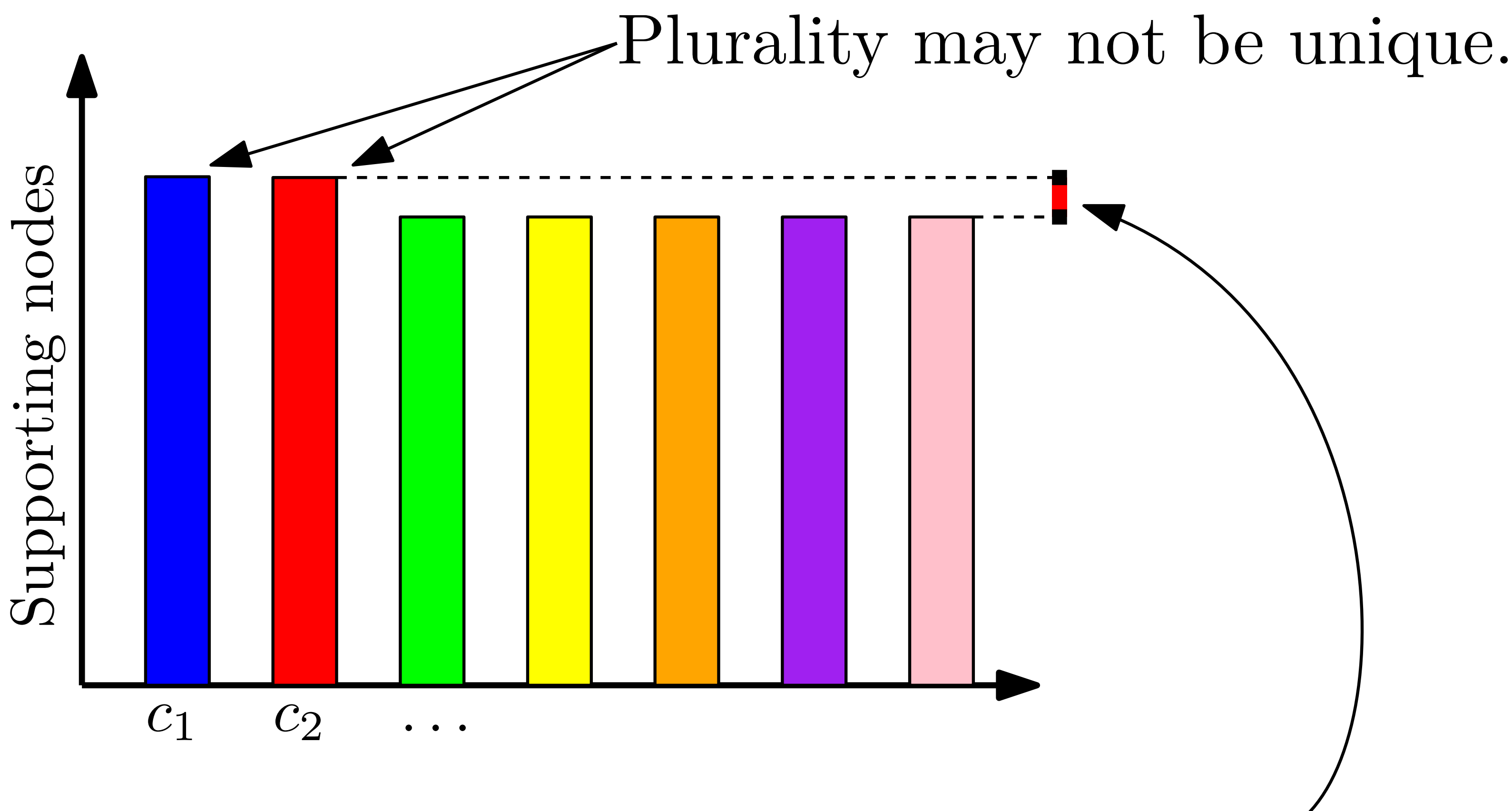
- First solution of the almost-stabilizing consensus problem in the uniform gossip model.
- Closes open question on convergence of 3-majority for $|\Sigma| > 2$.

The Problem without Bias



Very small gap between the plurality colors and second colors: one of the second colors may become plurality.

Analysis of 3-Majority

$C_i^{(t)}$:= number of nodes supporting color i at round t.

$\mu_j(\mathbf{c}) = \mathbf{E}[C_j^{(t+1)} \mid \mathbf{C}^{(t)} = \mathbf{c}]$

Lemma 1. For any color j it holds

$$\mu_j(\mathbf{c}) = c_j\left(1 + \frac{c_j}{n} - \frac{1}{n^2}\sum_{h\in[k]} c_h^2\right).$$

Lemma 2. Let 1 be a plurality color and j be a second-most-frequent color, then

$$\mu_1 - \mu_j \geqslant s(\mathbf{c})\left(1 + \frac{c_1}{n}\left(1 - \frac{c_1}{n}\right)\right).$$

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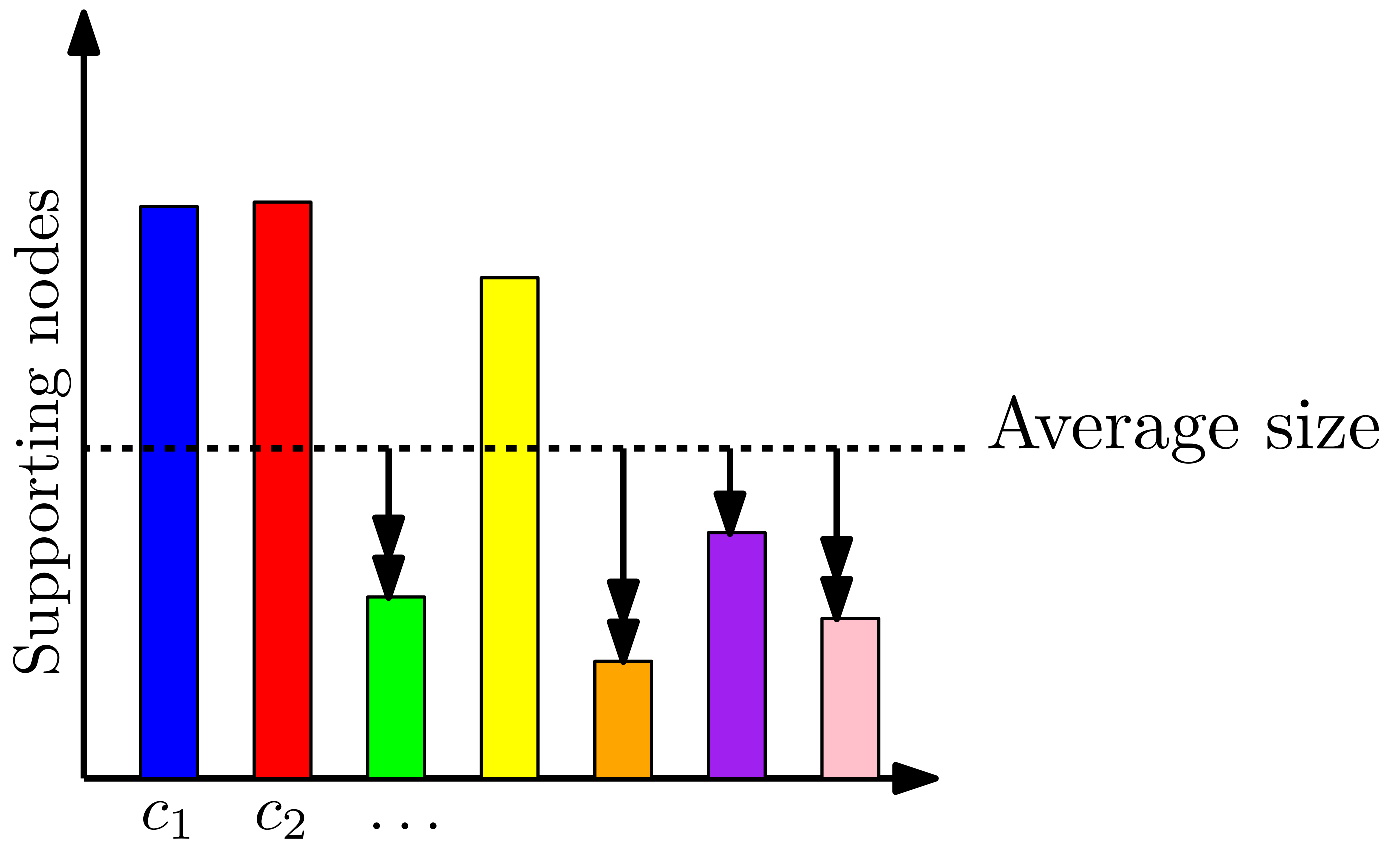
Indices are random variables: without bias, cannot concentrate on who is the plurality.

New Approach: Minorities Disappear

Lemma. Let $\mathbf{c}$ be the conf. at round t with j supported colors. For any color i it holds,

$$\mathbf{E}[C_i^{(t+1)} \mid \mathbf{C}^{(t)} = \mathbf{c}] \leq c_i \left(1 + \frac{c_i}{n} - \frac{1}{j}\right).$$



A “dying phase”

Lemma. Let $\mathbf{c}$ be any conf. with $j \leq n^{1/3-\varepsilon}$ supported colors ($\forall \varepsilon > 0$ const), and such that an color i exists with $c_i \leq n/j - \sqrt{jn \log n}$. Within $t = \mathcal{O}(j \log n)$ rounds color i becomes $\mathcal{O}\left(j^2 \log n\right)$ w.h.p.

$$c_i \leq n/j - \sqrt{jn \log n} \xrightarrow[\text{w.h.p.}]{t = \mathcal{O}(j \log n)} c_i = \mathcal{O}(j^2 \log n)$$

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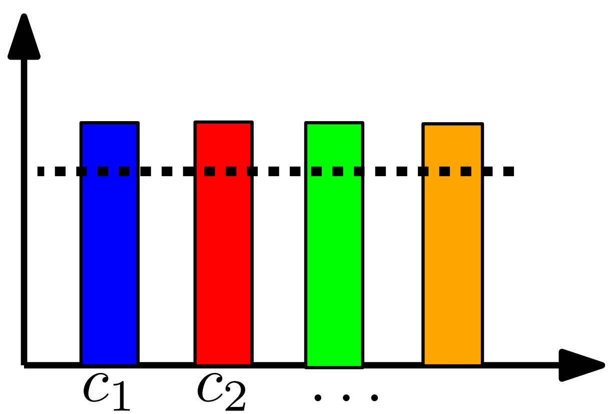
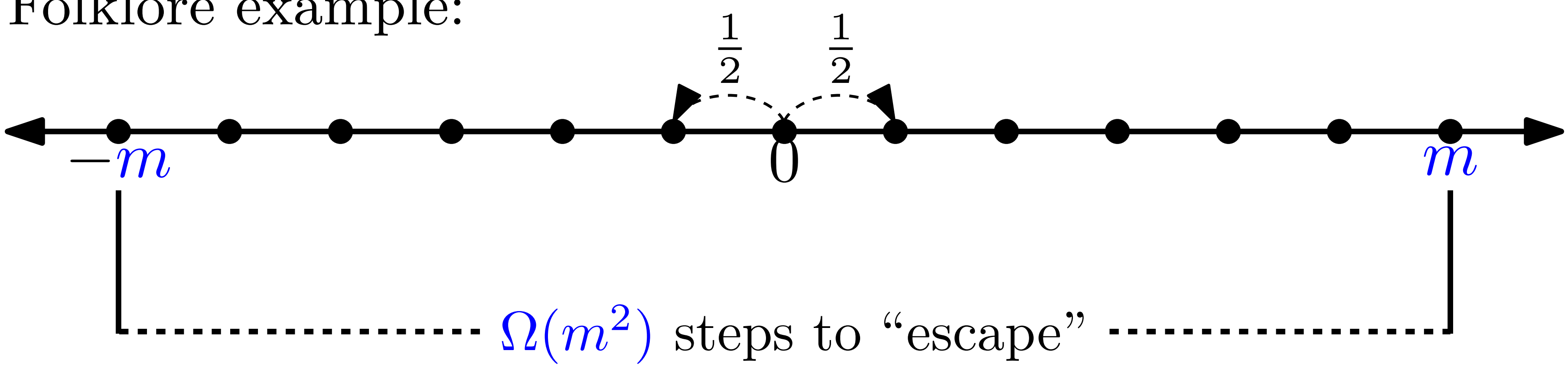
How to reach such imbalance from any configuration?



Simmetry Breaking

Folklore example:



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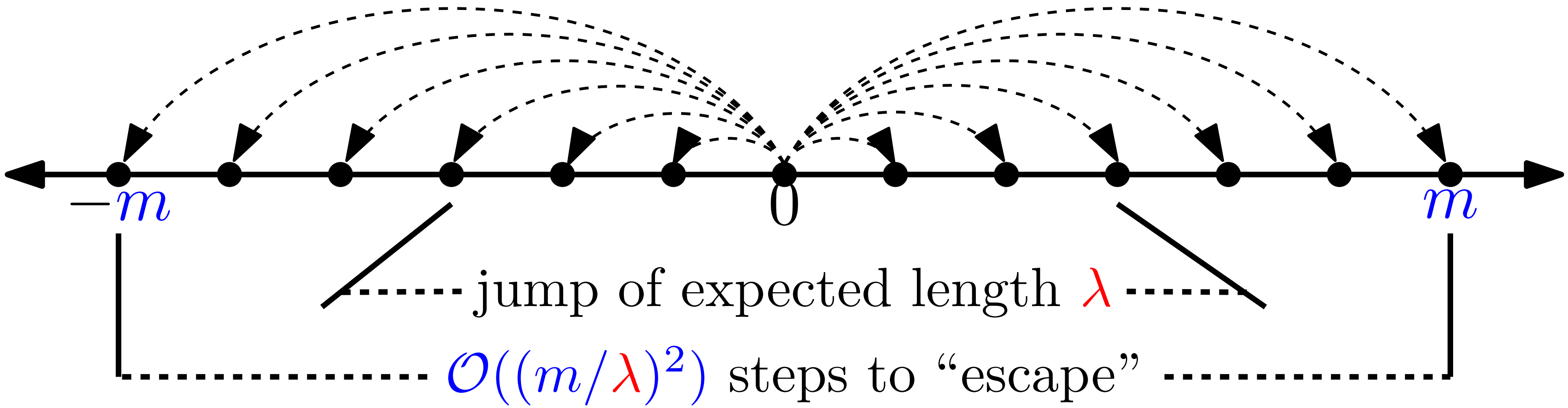

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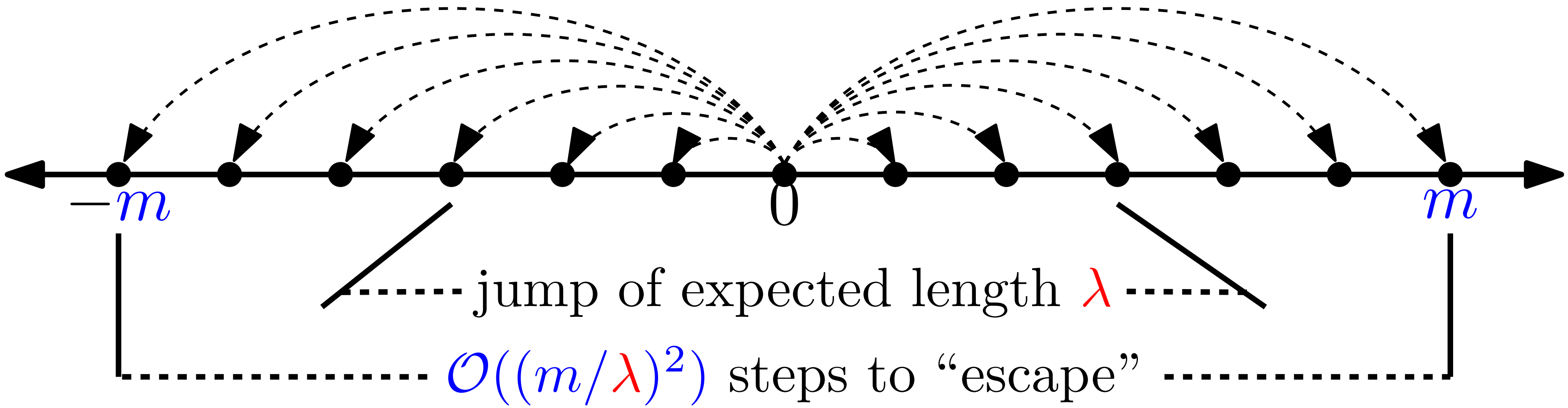


Lemma 42.
$\{X_t\}_t$ a Markov chain with finite state space Ω,
$f : \Omega \to \mathbf{N}$, $Y_t = f(X_t)$,
$m \in [n]$ a “target value” and $\tau = \inf\{t \in \mathbb{N} : Y_t \geq m\}$.
If $\forall x \in \Omega$ with $f(x) \leq m - 1$, it holds

1. *Positive drift*: $\mathbf{E}[Y_{t+1} \mid X_t = x] \geqslant f(x) + \lambda$ $(\lambda > 0)$,
2. *Bounded jumps*: $\Pr\{Y_\tau \geq \alpha m\} \leq \alpha m/n$, $(\alpha > 1)$,

then

$$\mathbf{E}[\tau] \leq 2\alpha \frac{m}{\lambda}.$$

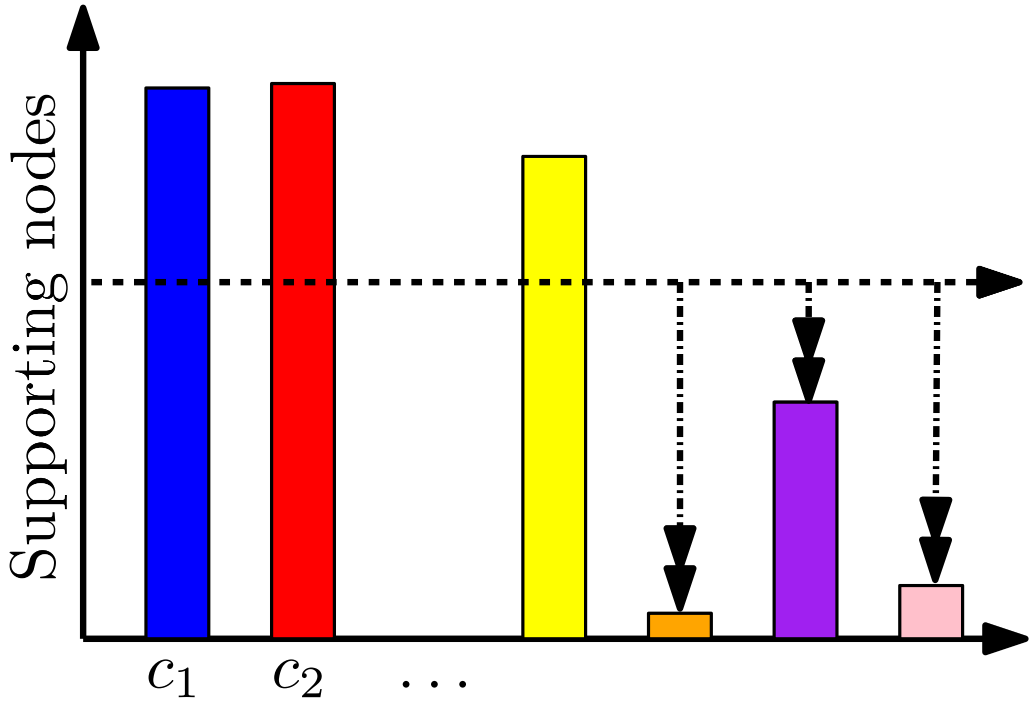
Simmetry Breaking

Lemma. Let $\mathbf{c}$ be any configuration with j supported colors. Within $t = \mathcal{O}\left(j^2\sqrt{\log n}\right)$ rounds it holds that

$$\Pr(\exists i \text{ such that } C_i^{(t)} \leq n/j - \sqrt{jn\log n}) \geq \frac{1}{2}$$

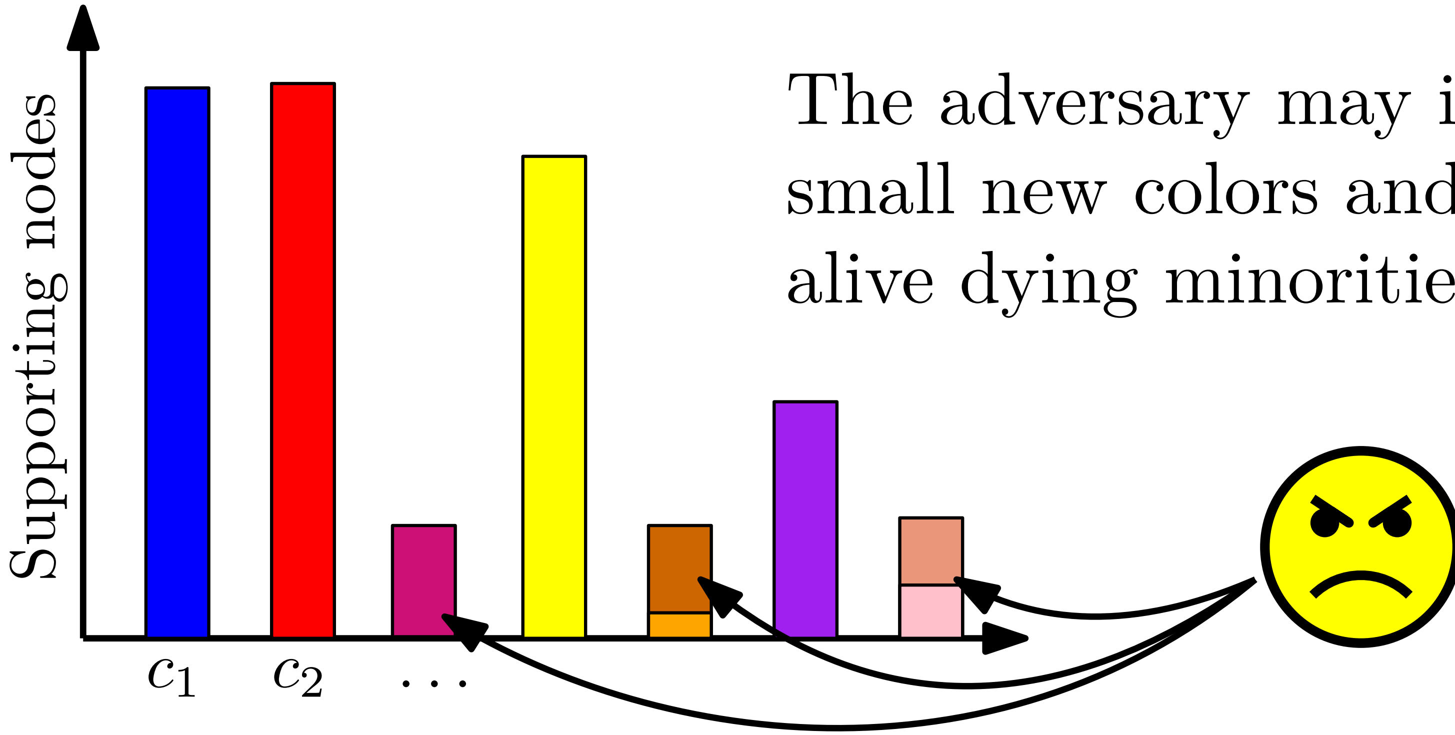
Proof. Let $\mathbf{m}(t)$ be the *index of minimum-size color* and apply Lemma 42 with $f(\mathbf{c}) = C_{\mathbf{m}(t)}$.

Handling the Adversary



Handling the Adversary

The adversary may introduce small new colors and keep alive dying minorities.



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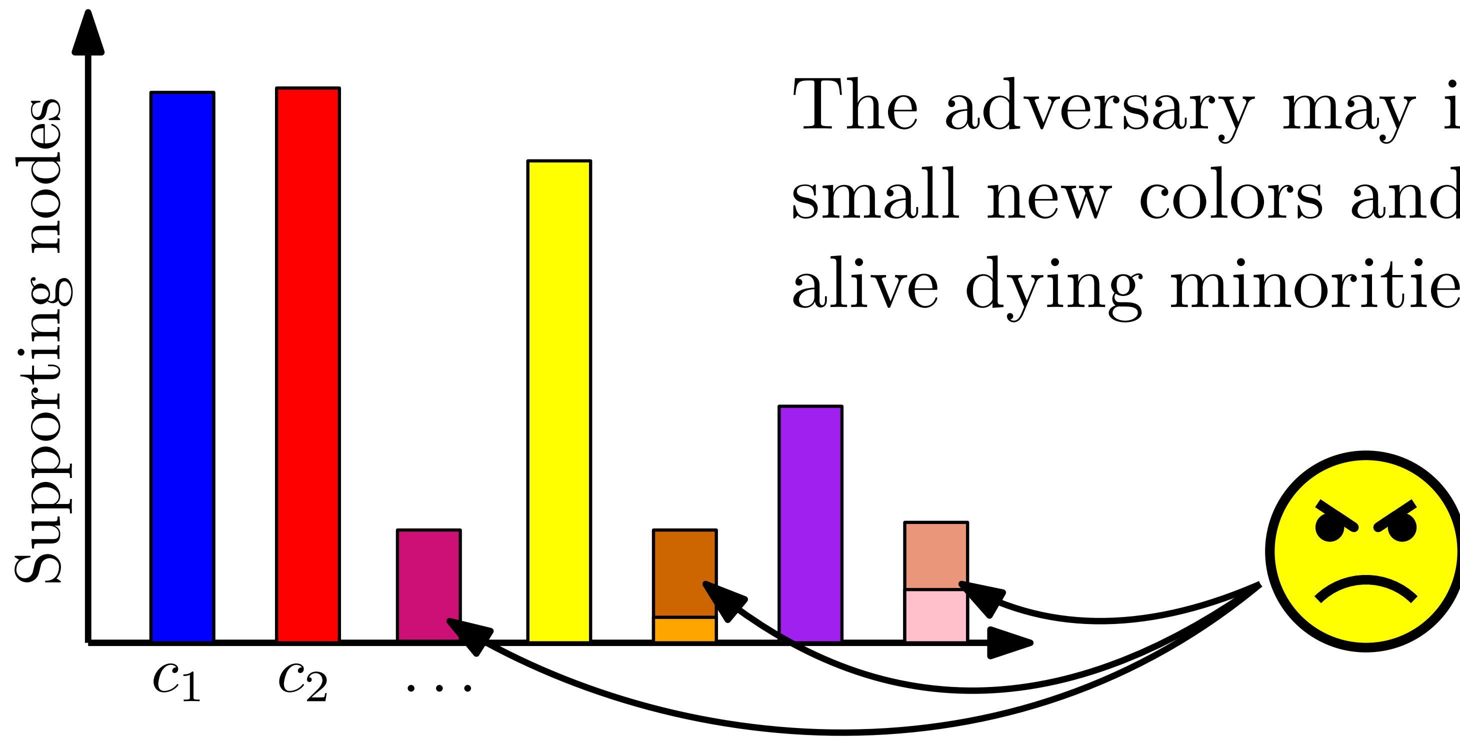

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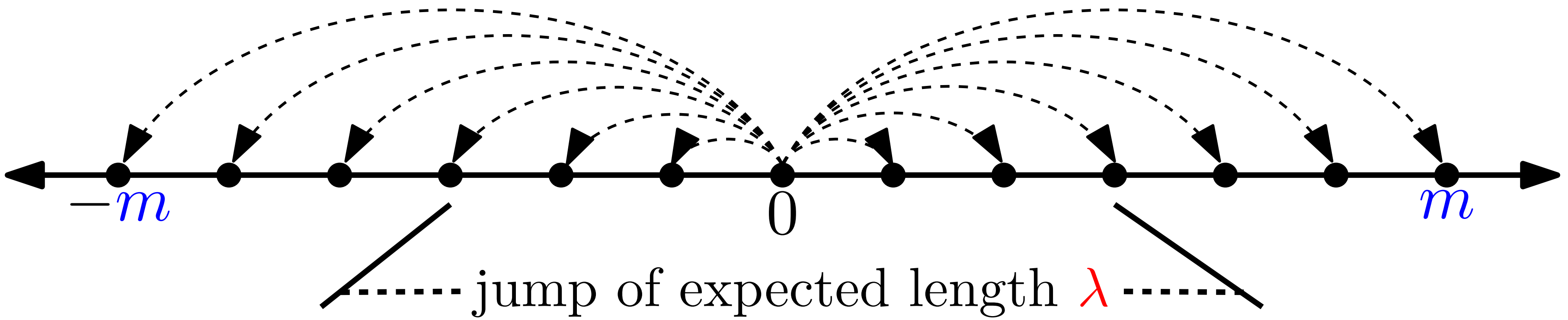
Supported colors

$$\mathbf{E}[\tilde{C}_i^{(t+1)} \mid \mathbf{C}^{(t)} = \mathbf{c}] \leq c_i \left(1 + \frac{c_i}{n} - \frac{1}{j} + \sqrt{\frac{k}{n}}\right)$$

$F = \mathcal{O}\left(\frac{\sqrt{n}}{(k^{\frac{5}{2}} \log n)}\right)$

“Big” colors

Handling the Adversary



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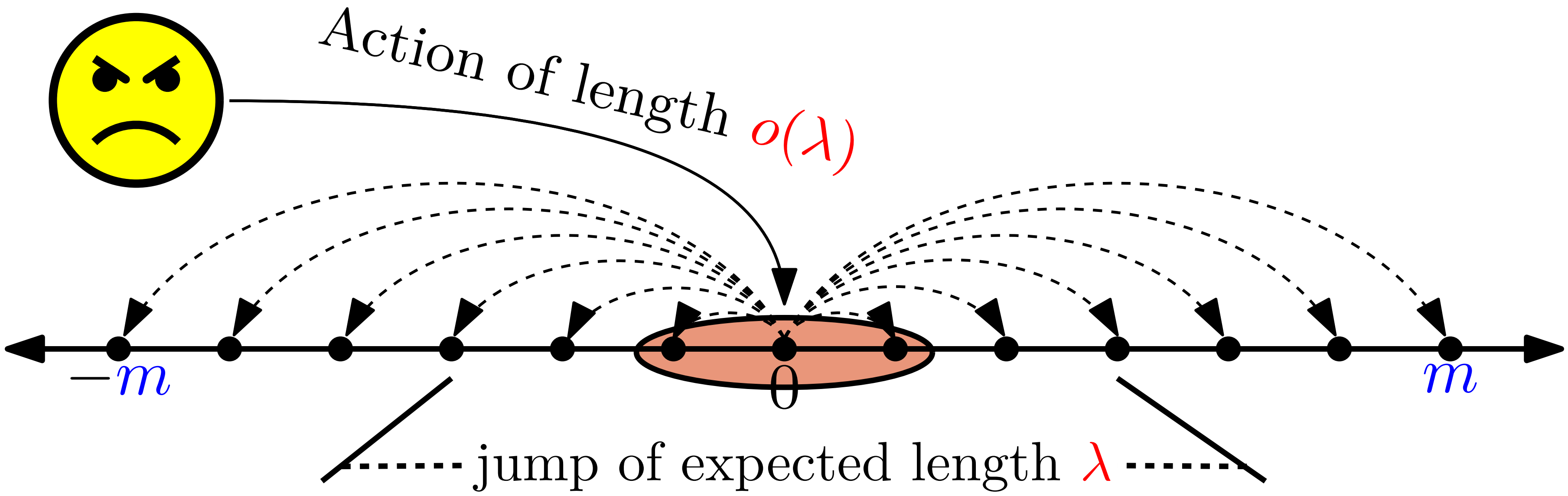

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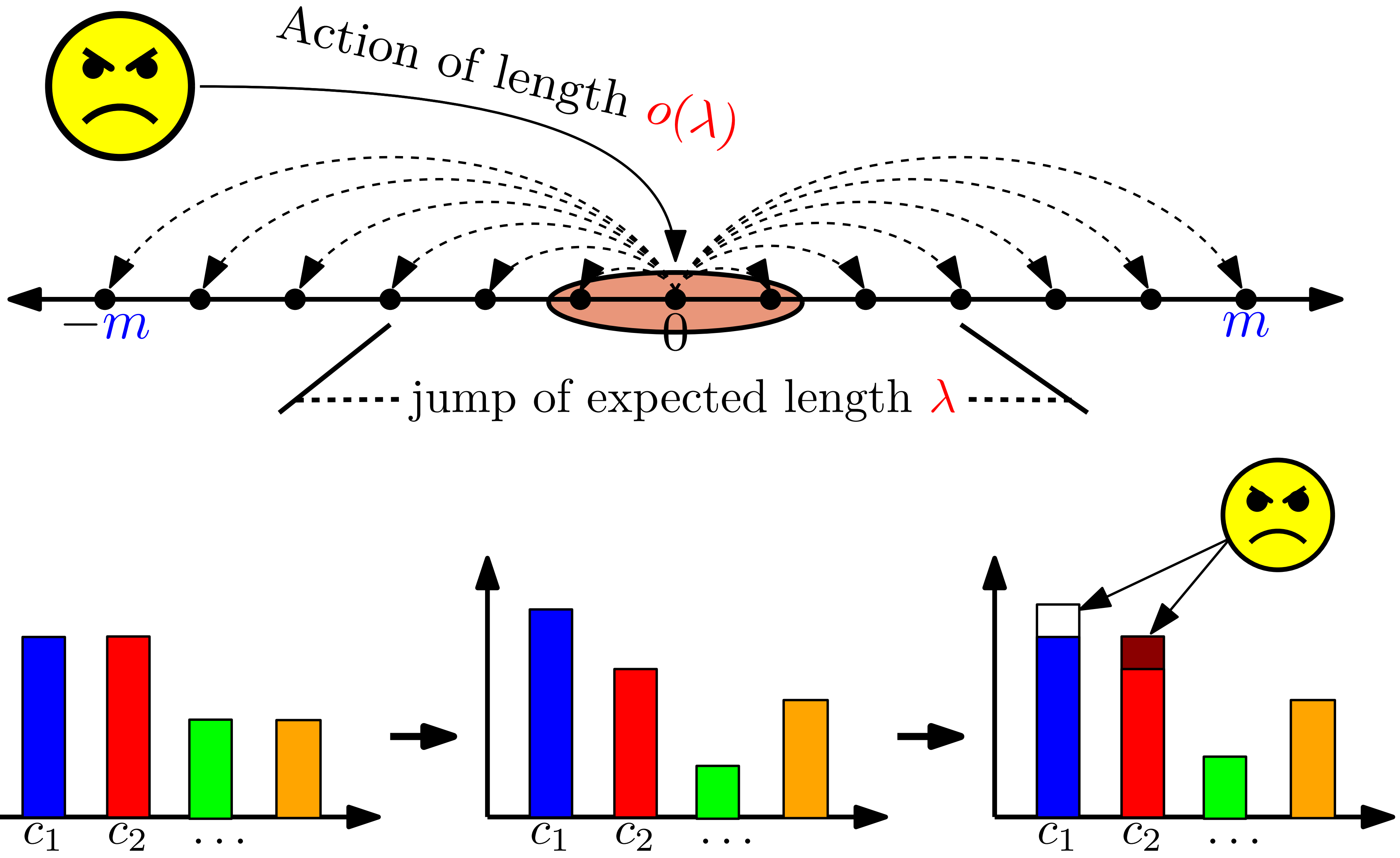

Open Problems

- Convergence in time $\mathcal{O}(k \log n)$?

- Stabilizing consensus on random/expander graphs?

Thank you!